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Mathematical Programming for Regional Water Quality Management

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U.S. DEPARTMENT OF THE INTERIOR • FEDERAL WATER QUALITY ADMINISTRATION

Mathematical Programming
for
Regional Water Quality Management

by

Graduate School of Business Administration
University of California
Los Angeles, California 90024

for the

FEDERAL WATER QUALITY ADMINISTRATION

DEPARTMENT OF THE INTERIOR

Program #16110 FPX
Formerly WP-01210

August, 1970

For sale by the Superintendent of Documents, U.S. Government Printing Office
Washington, D.C. 20402 - Price $1.25

WATER POLLUTION CONTROL RESEARCH SERIES

The Water Pollution Control Research Reports describe the results and progress in the control and abatement of pollution in our Nation's waters. They provide a central source of information on the research, development, and demonstration activities in the Federal Water Quality Administration, in the U. S. Department of the Interior, through inhouse research and grants and contracts with Federal, State, and local agencies, research institutions, and industrial organizations.

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ABSTRACT

This report is an application of a non-linear programming algorithm to the problem of optimal water quality control in an estuary. The mathematical model that was developed gives the solution to the general mixed case of at-source treatment, regional treatment plants, and by-pass piping. The non-linear algorithm is developed in considerable detail and a sample problem is worked out. Detailed results are presented for a pilot problem to illustrate the method of solution. Actual data from the Delaware Estuary is used to solve a large scale problem and the solution is given. The results indicate that a regional treatment system for the Delaware Estuary is superior, in terms, of total cost, to other proposed schemes. This report was submitted in fulfillment of Grant No. 16110 FPX between the Federal Water Quality Administration and the University of California, Los Angeles, directed by Dr. Glenn W. Graves, assisted by Andrew B. Whinston and Gordon B. Hatfield.

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CONCLUSIONS AND RECOMMENDATIONS

The Delaware Estuary provided an excellent test for our general non-linear mathematical programming model. At the optimal solution the three alternative treatment modes, treatment at the polluter, by-pass piping, and treatment at regional plants were all utilized. This would indicate that none of the treatment modes dominates the others in a heavily polluted estuary. Our final solution annual cost of 2.291 million dollars, to achieve a 3 mg/l dissolved oxygen standard throughout the estuary, compares very favorably with the 4.1 million annual cost of optimal treatment at the polluter, the next best known solution. The currently favored policy of requiring "uniform treatment" at all polluters would require an annual expenditure of 8.153 million to achieve the 3 mg/l quality standard. The solution time of about ten minutes for the complete model makes it practical to repeatedly solve the problem with various plant and pipe configurations, various quality goals and different cost estimates. It would seem, therefore, in terms of both utility and practicality, the non-linear programming model and solution techniques developed here provide an effective management tool in pollution control.

The difficulties encountered in solving the non-linear programming model were not due to the sheer size of over 2,000 variables and 80 constraints alone. The range of values of the transfer coefficients made the scaling of the numbers for single precision (6 decimal digits) calculation quite sensitive and compounded the usual numerical analysis difficulty of round-off error propagation. The economies of scale (mathematical concavity) inherent in piping costs and treatment plant costs also proved troublesome. In fact the opening and closing of regional plants and pipes hinges on very delicate physical considerations made in evaluating indeterminate cases in the first partial derivatives of the cost function. Many innovations in non-linear programming solution technique were developed and incorporated in the machine code to achieve the results and solution time reported here. The technique of only generating columns of the underlying local linear programming problems as needed, in addition to substantially reducing computation time as a direct effect, indirectly contributed to the effectiveness of the other innovations. The use of priority classes for the variables helps convergence, cuts computations, permits the solution of specialized sub-problems such as pure treatment at the polluter, and makes possible the opening of plants and pipes in spite of their concave cost functions. The use of logical techniques for range restricted variables and piecewise linear cost functions instead of expanding the size of the problem again substantially cuts computation time. Perhaps, the most significant innovation, however, was the use of a quantified "relaxation" technique. Instead of obtaining a complete optimal solution to each local linear programming problem in the non-linear step-wise procedure, a control parameter determines the degree of optimization. The most significant effect of this procedure is to lift the classical curse of "zig-zagging" in gradient optimization procedures. In addition, it permits decisive control of round-off error propagation.

In view of the successes achieved to date, we would recommend proceeding with further development of the approach. In particular, development of a "global" approach to deal with the combinatorial problem of treatment plant location and the non-convexity of the cost functions. Also a coupling of the present techniques with an up-dating procedure for the transfer coefficients which describe the physical quality response of the estuary would contribute to the realism of the model.

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I. INTRODUCTION

It has become apparent that major changes are desirable in the institutional structure for water quality management in the United States. The desirable structure would be a regional or basin-wide water quality management authority [1, p. 75].

A single agency would control all discharges (industrial and municipal) and operate all treatment plants in a region. It would construct new regional treatment plants in optimum locations and control the distribution and re-distribution of treated and partially treated wastes. The authority would be responsible for finding and implementing the least cost solution of meeting the stream (or estuary) water quality goals. The question of setting water quality goals relates to social and economic desires and needs. This question need not be resolved by the authority but it could make valuable contributions to the rationality of the process. When a change in goals is proposed, the authority will determine the optimum solution to meet the changes in addition to the cost. Thus, an informed public should be better able to decide what quality of water it is willing to pay for.

It is the purpose of this work to present a planning tool (algorithm) to provide optimal solutions for the complex choices involved in balancing alternative methods for attaining water quality goals. As one might suspect, there are a tremendous number of alternatives that would achieve these desired goals in a body of water. One of the most commonly proposed solutions is for the polluters to increase their levels of treatment. This is also one of the most expensive solutions.

Within the framework of a proposed regional water quality management authority, we are going to investigate this problem with some powerful tools from non-linear programming and control theory. Note that for the Delaware estuary, which we are going to study, an authority such as proposed above (namely the Delaware River Basin Commission) already exists [2].

It appears that Thomann [3] was the first to recognize the significance and usefulness of characterizing the dissolved oxygen changes in a body of water as a linear feedback and control system. Other models for predicting water quality in an estuary by solving the basic differential equations subject to boundary conditions have been presented [4]. However, the flexibility of a linear control approach appears to be far superior. This is due in part to the availability of an existing body of knowledge and in part to the tremendous amount of detail that can be brought to bear on the problem with no loss of computational effectiveness.

The concept of a transfer function is basic to every feedback and control system [5, 6]. Thomann has developed the transfer functions for a general body of water and in particular a one-dimensional estuary. In conventional feedback control system design, lead, lag, or an amplifier is introduced into the feedforward loop to modify the gain and re-orient the poles of the open loop transfer function. For the purposes of water quality control, the important thing is to obtain the transfer functions of the body of water. Control will be exercised not by modifying the transfer functions but by changing the input pattern to the control system.

The implementation of regional water quality control systems for the major rivers and estuaries of this country should occur sometime in the next 10 to 30 years. The implementation will be preceded, of course, by planning, analysis, and design of the system. At this point it is appropriate to give some guidelines to the governmental organizations or other agencies which will undertake these projects.

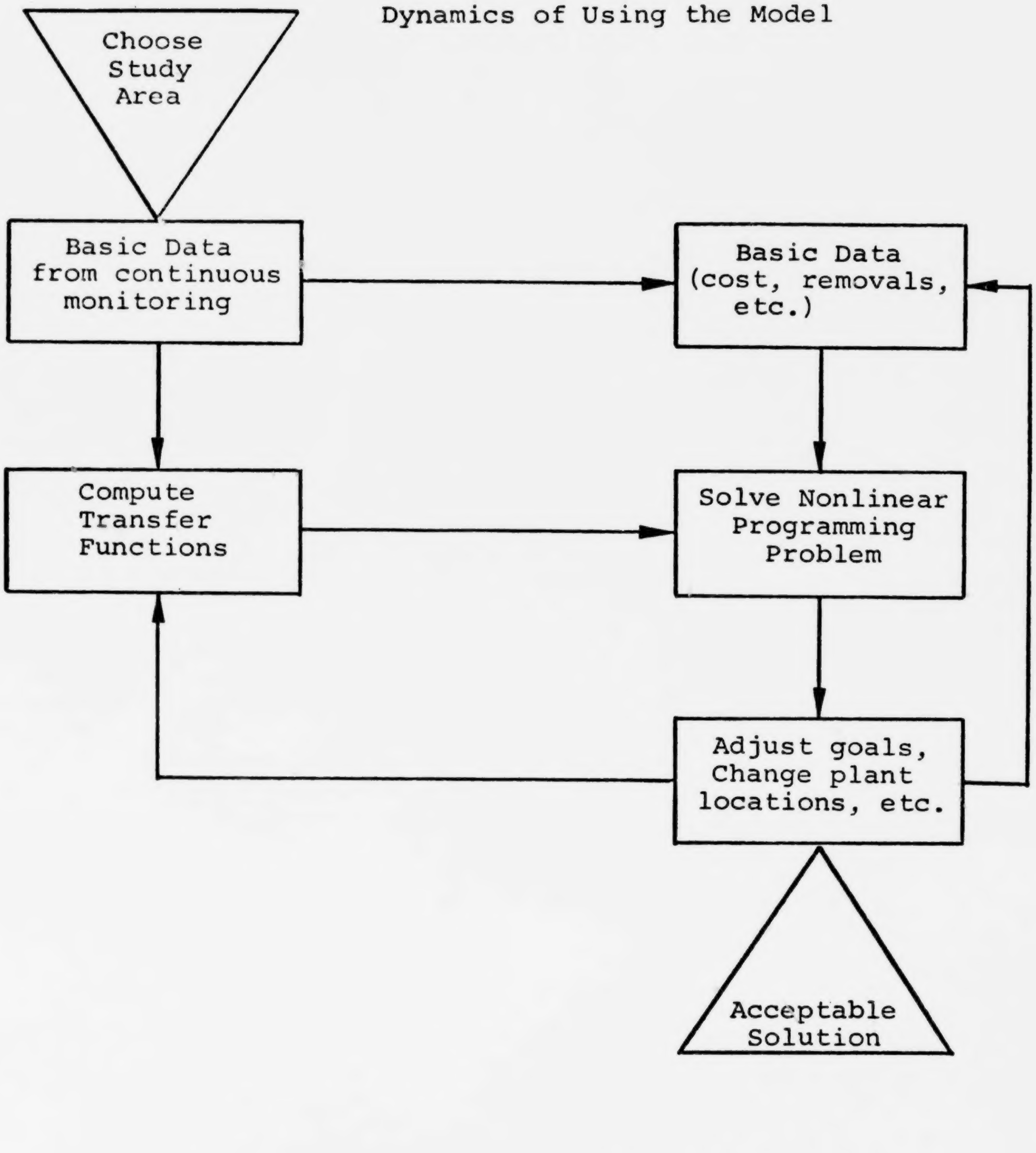
The most important consideration of the agency in charge of the project will be the overall approach to the problem. The methods used must be flexible enough and general enough so that in analyzing the physical system all relevant inputs are considered and accurate outputs are produced. Reality should not be compromised due to inadequate methods. Likewise, in the choice of control measures, the mathematical model must be of sufficient generality so that all possible alternatives are considered. It follows that the mathematical algorithm must be flexible enough to select from the tremendous number of possible solutions the "best" one in whatever sense "best" is taken.

At the present time, the approach which seems to adequately meet all these requirements is a combination of a linear feedback and control system and a nonlinear programming problem. The approach is general, flexible, and comprehensive. Any polluted body of water can be analyzed and any particular pollutant can be considered. Thomann gives extensive results for the variation of dissolved oxygen, DO, in a one-dimensional estuary. The choice of control measures includes treatment at polluter, treatment at regional plants, and by-pass piping taken singularly or in any combination.

A rough flow chart of the type of operations involved in implementing the overall approach is given in Figure 1. The remainder of this section will discuss the details involved in each of these operations.

FIGURE 1

Dynamics of Using the Model


Choose Study Area
Basic Data from continuous monitoring
Basic Data (cost, removals, etc.)
Compute Transfer Functions
Solve Nonlinear Programming Problem
Adjust goals, Change plant locations, etc.
Acceptable Solution

The considerations involved in determining the boundaries of the particular river or estuary are as follows: 1) all polluted sections of the body of water should be included; and 2) all sections, polluted or unpolluted, into which domestic or industrial waste is discharged should be included. Generally, the upstream boundary should be chosen at a location relatively unpolluted and the downstream boundary at a location where the recovery from pollution is complete (if such a location exists). For estuaries these locations might be the head of tide and the mouth of the bay. In the following discussion, we will consider the operations of Figure 1 that are relevant to an estuary where the water quality variable of interest is the dissolved oxygen.

The success of a linear systems approach depends on the availability of continuously recorded time series data. Water temperature, stream BOD, and dissolved oxygen should be continuously recorded at a number of monitoring stations along the estuary. This data is used in constructing the forcing functions and in evaluating the theoretical DO time series calculated from the transfer function. At the least, 1 year of record should be available before computing the transfer functions.

To calculate the transfer coefficients, the estuary is segmented into a finite number of sections. The data required for each section consists of:

1) rate of flow
2) volume and length of the section
3) the diffusion coefficient
4) reaeration coefficient

Also required are the flows and BOD concentrations for each of the polluters. This data is also basic data for the nonlinear programming problem. Specific equations for calculating the transfer functions of an estuary are given by Thomann.

The input to the nonlinear programming problem is the transfer functions and certain basic data. The data required for each polluter consists of:

1) rate of waste flow
2) BOD concentration
3) % removal of BOD
4) cost of removing BOD (linear piecewise costs)

The data required for a regional plant consists of:

1) performance curves (BOD removal efficiencies for different types of wastes separated and mixed)
2) cost curves for removal of BOD

Also required are pipeline costs and a preliminary set of DO goals.

Given a preliminary set of dissolved oxygen goals, initial computer runs will be made (using rough cost equations which we will supply) to determine the general pattern of the solution. This will identify preliminary regional treatment plant locations and preliminary pipeline locations. Surveys can be made in these areas to determine accurate piping costs and small-scale pilot plants can be built to determine the feasibility of mixing wastes, and to establish treatment efficiencies and costs. As this accurate data becomes available, further computer runs can be made which may change the solution and identify the need for better data in other areas. In addition, some re-calculation of the transfer functions may be required. This can possibly occur when large quantities of flow are shifted to different discharge points. Thus, we envision the interaction of lab teams, field teams, planners, and goal setters whose activities will be governed by the information generated by the program.

The entire process can be considered as a learning device. Each cycle of analyzing output from a computer run followed by changing DO goals and regional treatment plant locations, or restricting certain classes of variables to be active or non-active yields a certain amount of information about the response of the system. This information, properly interpreted, generates further actions and the cycle is repeated. This process is continued until an acceptable solution is obtained.

Considering the size and complexity of the problem, the cost of computer time, and assuming a reasonable number of computer runs, one of the prime goals of this work was to develop an algorithm that gives efficient solutions in a reasonable time. This became a formidable undertaking because of the many undesirable features inherent in our non-linear programming model of the physical situation. It has over 2000 variables and over 80 constraints. Some of the first partial derivatives are discontinuous and the transfer functions have such a wide range that scaling is extremely difficult.

Most of the techniques to be presented here have not been available to researchers attempting to solve large scale water pollution control problems. This is due to the rapid changes in fields outside water resources engineering and the limited contact with these fields by researchers. On this account, we present considerable detail on our solution techniques, particularly in the area of non-linear programming. It is, of course, not necessary to master the mathematics of these techniques as presented especially in Sections II and IV to evaluate the realism of the general mathematical model and the conclusions drawn from the model.

II. MATHEMATICAL DEVELOPMENT

The mathematical development of this section will consist of a proof of convergence of the general non-linear programming algorithm followed by a sample problem worked out in complete detail. A previous paper by the authors [7, pages 36-47] gave a complete discussion of the solution of large scale linear programming problems. The idea given there was to reduce the number of calculations by carrying a truncated tableau and generating elements as needed. This approach is appropriate for problems where the number of variables is much larger than the number of constraints, or vice versa. It was shown that, by carrying only the matrix inverse of the currently binding constraints, any element (or column) of the basic tableau could be generated from a few simple equations. This, of course, assumes that the elements of the underlying array are available by generation or from combinatorial considerations. Thus up-dating the truncated tableau instead of the complete basic tableau greatly reduces the number of calculations.

In the discussion that follows, it will be shown that for every non-linear programming problem we can define an associated local linear programming problem. In fact most of the burden of calculation in solving a non-linear problem is in solving the series of local linear problems as we move to successive points. Thus we can appreciate the efficiencies obtained by using the truncated tableau to solve the local L.P.'s of the non-linear problem.

Development of a non-linear programming algorithm

The algorithm employed for solving the non-linear programming problems of this paper is a general purpose algorithm which solves problems of the form:

$$\begin{aligned} &\text{Subject to:} \quad g^i(y) \leq 0 \qquad i = 1,\ldots,m-1 \\ &\qquad\qquad \min_{y} \quad g^m(y) \end{aligned} \tag{1}$$

where y is a vector in E^n and $g^i(y)$, $i = 1,\ldots,m$, are differentiable functions. It is a "local", "gradient", "stepwise" correction descent algorithm. By a "stepwise" procedure we mean given a point y^o in the domain of the functions, a "correction" vector Δy is determined and a new point $y = y^o + k\,\Delta y$ is used for the successor "step". It is a "local" method because the correction direction Δy and its length (determined by the scalar k) are obtained from the behavior of the system in a "sufficiently" small neighborhood of the current point y^o. It is a "gradient" technique inasmuch as the gradients of the function $g^i(y)$ are principally used to obtain the correction direction.

The algorithm is a "local-gradient" technique because of reliance on the approximation of the functions $g^i(y)$ by the well known:

Approximation Theorem

Let $g \epsilon C'$ in an open region D and let E be a closed bounded subset of D. Let

$$\nabla g(y^o) = \left[\frac{\partial g(y^o)}{\partial y_1}, \frac{\partial g(y^o)}{\partial y_2}, \ldots, \frac{\partial g(y^o)}{\partial y_n}\right]$$

be the "gradient" of g at the point $y^o \epsilon E$. Then,

$$g(y^o + \Delta y) = g(y^o) + \nabla g(y^o) \cdot \Delta y + R(\Delta y) \tag{2}$$

where

$$\lim_{\Delta y \to 0} \frac{R(\Delta y)}{|\Delta y|} = 0$$

uniformly for $y^o \epsilon E$.

The direction of improvement Δy in the stepwise procedure is obtained from the function approximation in (2) by estimating the $R^i(\Delta y)$ and solving the associated local linear programming problem:

$$\nabla g^i(y^o) \cdot \Delta y \leq -g^i(y^o) - \bar{k}\, r^i \tag{3}$$

$$\min \nabla g^m(y^o) \cdot \Delta y$$

The r^i are an arbitrary set of positive constants which serve as surrogates for the $R^i(\Delta y)$, where $r^i = 0$ for the strictly linear functions. Of course the $R^i(\Delta y)$ are truly functions of the direction Δy being sought, and substituting positive constants causes the solution of the local linear programming problem to be an approximation. In practice, the r^i are taken to be the $R^i(\Delta y)$ obtained from the previous Δy correction vector. The $\bar{k}$ is used to parametrically adjust the solutions of the local linear programming problems. In general as $\bar{k}$ decreases, it becomes easier to achieve feasibility in the local linear programs but unfortunately the gain that can be assured in the convergence of the non-linear problem also decreases. As will be shown in the convergence proof it is necessary to have $\bar{k} \geq s > 0$ for some fixed s to prevent convergence to a non-optimal limit point.

NATURE OF THE SOLUTION

In order to discuss the convergence of the algorithm we must develop the concept of an ε-solution of the problem. Given any $\varepsilon > 0$, it is first necessary to construct a suitable "feasible" set. This consists of exhibiting an ε_1 such that $0 < \varepsilon_1 \leq \varepsilon$ and defining the ε-feasible set $F = \{y \mid y \in E$ and $g^i(y) \leq \varepsilon_1$, $i=1,\ldots,m-1\}$. The solutions y^* are then characterized as:

Global ε-minimums

For all $y \in F$

$$g^m(y^*) \leq g^m(y) + \varepsilon$$

or

Local ϵ-minimums

For some $\delta > 0$ and all $y \epsilon F$ such that

$$|y-y^*| < \delta$$

$$g^m(y^*) \leq g(y) + \epsilon$$

Obviously the concept of an ϵ-solution is necessary when treating real valued functions. The solutions in general will be real valued numbers and an answer can be determined only when the level of accuracy desired is specified. A problem is considered completely solved when for an arbitrary positive ϵ an ϵ-solution can be guaranteed in a finite number of steps.

We also require the following well known fundamental results from the theory of linear programming. A general mixed linear programming problem can always be stated in both a primal and dual form.

Primal Form

Subject to the constraints

$$\sum_q a_{iq}\, y_q = a_{in};\quad 1 \le i \le m_1$$

$$\sum_q a_{iq}\, y_q \le a_{in};\quad m_1 < i < m \qquad (6)$$

$$y_q \ge 0 \qquad n_1 < q < n$$

Minimize

$$\sum_q a_{mq}\, y_q$$

Dual Form

Subject to the constraints

$$\sum_p x_p a_{pj} = a_{mj};\quad 1 \le j \le n_1$$

$$\sum_p x_p a_{pj} \le a_{mj};\quad n_1 < j < n \qquad (7)$$

$$x_p \le 0 \qquad m_1 < p < m$$

Maximize

$$\sum_p x_p a_{pn}$$

The fundamental inequality of linear programming states:

> Given any feasible solution y_q of the constraints in (6) and any feasible solution x_p of the constraints in (7), then
>
> $$\sum_q a_{mq}\, y_q \ge \sum_p x_p a_{pn} \qquad (8)$$

The solvability or unsolvability of the linear programming problem as well as the relationship between the primal and dual problems can be summarized in a general theorem.

Linear Programming Theorem

Given a linear programming problem, exactly one of the following conditions holds:

1. There exists a finite value v and feasible vectors y^*_q and x^*_p of (6) and (7) such that

$$\sum_q a_{mq}\, y^*_q = \sum_p x^*_p\, a_{pn} = v \tag{9}$$

   (and by the fundamental inequality (8), they must then be optimal feasible vectors).
2. The constraints for the primal problem are inconsistent, and the constraints of the dual problem are inconsistent or the dual extremal function is unbounded.
3. The primal extremal function is unbounded and the constraints for the dual problem are inconsistent.

There are three bounds required in our development.

Bound 1

$$\sum_p (-x_p) \le B_1 \tag{10}$$

The bound on the magnitude of the dual variables encountered in solving the local linear programming problems has to be imposed as an additional condition on the problem. This condition merely acknowledges a limit to our computational ability to distinguish non-singularity in inverting the basis associated with the local linear programming problem.

Bound 2

$$|\Delta y| \leq B_2 \tag{11}$$

Since E is a bounded set, the vectors y of interest are bounded. By the triangle inequality

$$|\Delta y| = |y - y^o| \leq |y| + |y^o|$$

and the correction vector is also bounded. Therefore, we can always superimpose constraints of the form $L^i \leq y_i \leq U^i$, which can be treated algebraically without explicit introduction into the constraint set. These bounds will always exclude alternative 3, an unbounded solution in solving the local linear programming problems.

Bound 3

$$|R^i(\Delta y)| \leq B_3, \{i = 1,\ldots,m\} \tag{12}$$

Transposing (2),

$$R^i(\Delta y) = g^i(y) - g^i(y^o) - \nabla g^i(y^o)\cdot\Delta y.$$

But since $g^i(y)\epsilon C'$, both $g^i(y)$ and $\nabla g^i(y^o)\cdot\Delta y$ are continuous functions and the difference of continuous functions is a continuous function. Hence $R^i(\Delta y)$ is a continuous function over a compact set E, and is therefore bounded. Taking the largest bound over the finite set of bounds for the $R^i(\Delta y)$, we obtain an over all bound for the remainder terms.

When the constraints of the primal problem are inconsistent in solving a local linear programming problem, there still exists a set of dual variables to a restricted subproblem of the original problem. These are exhibited naturally in the linear programming algorithm employed [see "A Complete Constructive Algorithm for the General Mixed Linear Programming Problem," Naval Research Logistics Quarterly, Vol. 12, No. 1, March 1965]. Although not all algorithms naturally exhibit these dual variables, their existence is easily demonstrated. Let H index a consistent subset

$$H = \{i \mid \nabla g^i(y^o)\cdot \Delta y \leq (-g^i(y^o) - \bar{k}\, r^i) \text{ for some } \Delta y\},$$

(where choosing $\Delta y = L_i$ the set H contains at least the bounding constraints of the form $L_i \leq \Delta y_i \leq u_i$), and w is an index such that

$$\nabla g^w(y^o)\cdot \Delta y \geq -g^w(y^o) - \bar{k}\, r^w.$$

Solve the subproblem

Subject to

$$\nabla g^i(y^o)\cdot \Delta y \leq -g^i(y^o) - \bar{k}\, r^i, \quad i\epsilon H$$

Minimize

$$\nabla g^w(y^o)\cdot \Delta y \tag{13}$$

In this subproblem only the first alternative can occur since we know the problem to be consistent and bounded. Therefore, we will always have a w and a set of dual variables $x_p \leq 0$ such that

$$\nabla g^w(y^o) = \sum_{p\epsilon H} x_p \nabla g^p(y^0) \tag{14}$$

(since the Δy are of arbitrary sign and the dual variables solve the dual constraints), and

$$\nabla g^w(y^o)\cdot \Delta y = \sum_{p\epsilon H} x_p(-g^p(y^o) - \bar{k}\, r^p) \tag{15}$$

(since only the first alternative of the Linear Programming Theorem can occur).

For a subproblem where $w < m$, to be consistent at its conclusion and employing (15), $\bar{k}$ is bounded above since,

$$\nabla g^w(y^o)\cdot\Delta y = \sum_{p\in H} x_p(-g^p(y^o) - \bar{k}\, r^p) \le -g^w(y^o) - \bar{k}\, r^w$$

or

$$\bar{k} \le -(g^w(y^o) + \sum_{p\in H} (-x_p)\cdot g^p(y^o)/(r^w + \sum_{p\in H} (-x_p) r^p). \qquad (16)$$

In actual practice the $\bar{k}$ is parametrically adjusted downwards to a lower bound as required in the course of the computation. A bound of

$$\bar{k} \ge \epsilon/(B_3 + B_1\cdot B_3) \qquad (17)$$

will be sufficient to insure a gain in the original non-linear problem. Using (16) and (17), the condition for a violation of this bound is,

$$\sum_{p\in H} (-x_p)\cdot g^p(y^o) \ge -g^w(y^o) - \epsilon \qquad (18)$$

In general, from (15), after reaching non-linear feasibility,

$$\nabla g^m(y^o)\cdot\Delta y = \sum_p (-x_p) g^p(y^o) + \bar{k}\cdot\sum_p (-x_p) r^p \qquad (19)$$

where

$$\sum_p (-x_p)\cdot g^p(y^o) \le 0$$

and (20)

$$\sum_p (-x_p) r^p > 0$$

Hence, a small $\bar{k}$ improves the rate of gain but decreases the admissible step size. The art of constructing an efficient algorithm requires the balancing of the rate of gain with the admissible step size.

We can now give the conditions for a stationary point of the algorithm.

Local Linear Stationary Point

There exists a set of dual variables x_p such that

1) $\sum_{p} (-x_p) > B_1$ (insufficient resolution)

There exists a w and H such that

2) $\sum_{p\epsilon H} (-x_p)\cdot g^p(y^o) \geq -g^w(y^o) - \epsilon$ (inconsistency)

There exists a $y_0 \epsilon F$ such that

3) $\sum_{i=1}^{m-1} (-x_i)\cdot g^i(y^o) \geq -\epsilon$ (optimality)

An interpretation of the consequences of termination of the algorithm with these stopping criteria will be provided after a demonstration that they are the only conclusions.

Convergence Theorem

It is always possible to choose $\bar{k}$ and k in such a manner that in at most a finite number of steps a local linear stationary point is reached.

Proof:

Choose $\bar{k} = \epsilon/(B_3 + B_1B_3)$.

In order to establish k, we must dominate the remainder terms $R^i(\Delta y)$ and require the following consequence of the Approximation Theorem (2):

For any $y^o \epsilon E$, there exists a $\delta^i > 0$ such that for $k\cdot|\Delta y| \leq \delta^i$,

$$R^i(k\cdot\Delta y)/k\cdot|\Delta y| < \frac{\bar{k}\ r^i}{B_2} \quad , \quad i = 1,\ldots,m-1 \qquad (21)$$

$$R^m(k\cdot\Delta y)/k\cdot|\Delta y| < \frac{\epsilon}{B_2} \cdot \frac{1}{2\cdot(1+B_1)} \qquad (22)$$

Recalling that $|\Delta y| \leq B_2$, and taking $\delta = \min_i \delta^i$, then for $k = \delta/B_2$,

$$R^i(k\cdot\Delta y)/k < \bar{k}\, r^i, \quad i = 1,\ldots,m-1 \tag{24}$$

$$R^m(k\cdot\Delta y) < \frac{\varepsilon\cdot\delta}{B_2}\cdot\frac{1}{2\cdot(1+B_1)} \tag{25}$$

Let

$$\text{supg} = \max_i \{g^i(y^o)\,|\,g^i(y^o) > 0\}$$

or (26)

$$\text{supg} = 0 \quad \text{if } g^i(y^o) \leq 0, \quad i = 1,\ldots,m-1$$

The feasibility of the constraints is then achieved or maintained when k is choosen to satisfy the constraints,

$$g^i(y) = g^i(y^o) + \nabla g^i(y^o)\cdot\Delta y\cdot k + R^i(k\cdot\Delta y) \leq d\cdot\text{supg} \tag{27}$$

with $0 \leq d < 1$, and as close to zero as possible when $\text{supg} > 0$.

Assume we do not encounter a local linear stationary point. At the conclusion of the solution of the local linear programming problem (3) we cannot be inconsistent without violating stationary condition 2). Thus,

$$\nabla g^i(y^o)\cdot\Delta y\cdot k \leq [-g^i(y^o) - \bar{k}\, r^i]\cdot k. \tag{28}$$

Substituting this into (27) it is sufficient to choose k such that the stronger condition holds:

$$g^i(y^o) + [-g^i(y^o) - \bar{k}\, r^i]\cdot k + R^i(k\cdot\Delta y) \leq d\cdot\text{supg} \tag{29}$$

or

$$-\bar{k}\, r^i \leq \frac{d\ \text{supg} - (1-k)\cdot g^i(y^o)}{k} - \frac{R^i(k\cdot\Delta y)}{k} \tag{30}$$

With an appropriate choice of d, the term

$$\frac{d\cdot \text{supg} - (1-k)\cdot g^i(y^o)}{k} \geq 0 \quad \text{for } 0 < k < 1$$

Choose $d = 0$ when supg $= 0$, inasmuch as supg $= 0$ implies $g^i(y^o) \leq 0$. Choose $d = (1 - k)$ when supg > 0, and recall supg $\geq g^i(y^o)$.

Now with our choice of $k = \delta/B_2$ and invoking (24), the condition (30) is satisfied at every step. But then when $d = (1 - k) = (1 - \delta/B_2)$, the infeasibility supg must be reduced by at least $\delta/B_2\cdot$supg at each step. Given $\epsilon_1 > 0$ in at most a finite number of steps, supg $< \epsilon_1$ and remains so in all subsequent steps.

Further, at the conclusion of the solution of the local linear programming problems, we must have,

$$\sum_p (-x_p)\cdot g^p(y^o) < -\epsilon \tag{31}$$

or we would violate condition 3 in our definition of a stationary point. Restating (19) for convenience,

$$\nabla g^m(y^o)\cdot\Delta y = \sum_p (-x_p)\cdot g^p(y^o) + \bar{k}\sum_p (-x_p)\cdot r^p. \tag{32}$$

Employing these results, we can demonstrate a bounded reduction in the extremal function as follows:

$$\begin{aligned} g^m(y) &= g^m(y^o) + \nabla g^m(y^o)\cdot\Delta y\cdot k + R^m(k\cdot\Delta y) \\ &\qquad\qquad [\text{by } (32)] \\ &= g^m(y^o) + [\sum_p(-x_p)\cdot g^p(y^o) + \bar{k}\cdot\sum_p(-x_p)\cdot r^p]\cdot k \\ &\qquad + R^m(k\cdot\Delta y) \\ &\qquad [\text{by } (31), \text{ our choice of } \bar{k} \text{ and } k, \text{ and } (25)] \\ &\leq g^m(y^o) + [-\epsilon + \frac{\epsilon}{(B_3+B_1\cdot B_3)}\cdot B_1\cdot B_3]\cdot\frac{\delta}{B_2} \\ &\qquad + \frac{\epsilon\cdot\delta}{B_2}\cdot\frac{1}{2\cdot(1+B_1)} \end{aligned}$$

[by algebraic simplification]

$$= g^m(y^o) - \frac{\epsilon\cdot\delta}{B_2}\cdot[1 - \frac{B_1}{(1 + B_1)} - \frac{1}{2\cdot(1 + B_1}]$$

$$= g^m(y^o) - \frac{\epsilon\cdot\delta}{2B_2(1 + B_1)} \qquad (33)$$

Inasmuch as $g^m(y)$ is reduced by at least $\epsilon\cdot\delta/2\cdot B_2(1+B_1)$ at each step, we can contradict any bound on $g^m(y)$ in a finite number of steps. Hence we conclude that a stationary point is reached in at most a finite number of steps.

In order to interpret a stationary point, we need an additional result. Consider any $y = y^0 + \Delta y$ that is an ϵ-feasible solution of the constraints of (1), the original problem. By the Approximation Theorem

$$g^p(y) = g^p(y^o) + \nabla g^p(y^o)\cdot\Delta y + R^p(\Delta y) \leq \epsilon_1$$

or

$$\nabla g^p(y^o)\cdot\Delta y \leq -g^p(y^o) - R^p(\Delta y) + \epsilon_1 \qquad (34)$$

Now recall that for any solution $x_p \leq 0$ of the dual problem as stated in (14),

$$\sum_p x_p \nabla g^p(y^o) = \nabla g^m(y^o). \qquad (35)$$

Using these two results, (34) and (35), we can obtain a lower bound on the extremal function as follows:

$$\begin{aligned} g^m(y) &= g^m(y^o) + \nabla g^m(y^o)\cdot\Delta y + R^m(\Delta y) \\ &= g^m(y^o) + [\sum_p x_p \nabla g^p(y^o)]\cdot\Delta y + R^m(\Delta y) \\ &= g^m(y^o) + \sum_p x_p [\nabla g^p(y^o)\cdot\Delta y] + R^m(\Delta y) \\ &\geq g^m(y^o) + \sum_p (-x_p)\, g^p(y^o) + \sum_p (-x_p)\cdot R^p(\Delta y) + R^m(\Delta y) \\ &\quad + \sum_p x_p\cdot\epsilon_1 \qquad (36) \end{aligned}$$

Choose $\epsilon_1 = \frac{\epsilon}{B_1}$ and assume that condition 1) of a local linear stationary is not violated then

$$\sum_p x_p \geq -B_1 \quad \text{and} \quad \sum_p x_p \cdot \epsilon_1 \geq -\epsilon \ .$$

Using this fact and recalling that global or local convexity for all functions would imply $R^i(\Delta y) \geq 0$. Formula (36) would then imply,

$$g^m(y) \geq g^m(y^o) + \sum_p (-x_p) \; g^p(y^o) - \epsilon \tag{37}$$

When condition 3) in the definition of a stationary point obtains, we conclude

$$g^m(y^o) \leq g^m(y) + 2\epsilon \quad \text{for all } y, \tag{38}$$

and therefore by definitions (4) and (5) we have achieved a global or local ϵ-minimum. The identical argument can be applied to a subproblem when condition 2) in the definition of a stationary point obtains, to demonstrate ϵ-inconsistency. In fact, if the stronger condition

$$\sum_{p \in H} (-x_p) \cdot g^p(y^o) > -g^w(y^o) \tag{39}$$

occurs at termination, the argument would demonstrate absolute inconsistency of the constraints. Without convexity it is theoretically possible to terminate at a "saddlepoint", although this is virtually impossible as a practical matter. The algorithm can be extended to a second order method by explicit introduction of first order necessary conditions as additional constraints. It can also be extended to treat equality constraints.

An Example Problem

As an illustration of some of these ideas consider the following non-linear programming problem.

Subject to:

$$g^1(y) = y_1^2 + y_2^2 - 9 \leq 0$$

$$g^2(y) = -y_1 - y_2 + 1 \leq 0$$

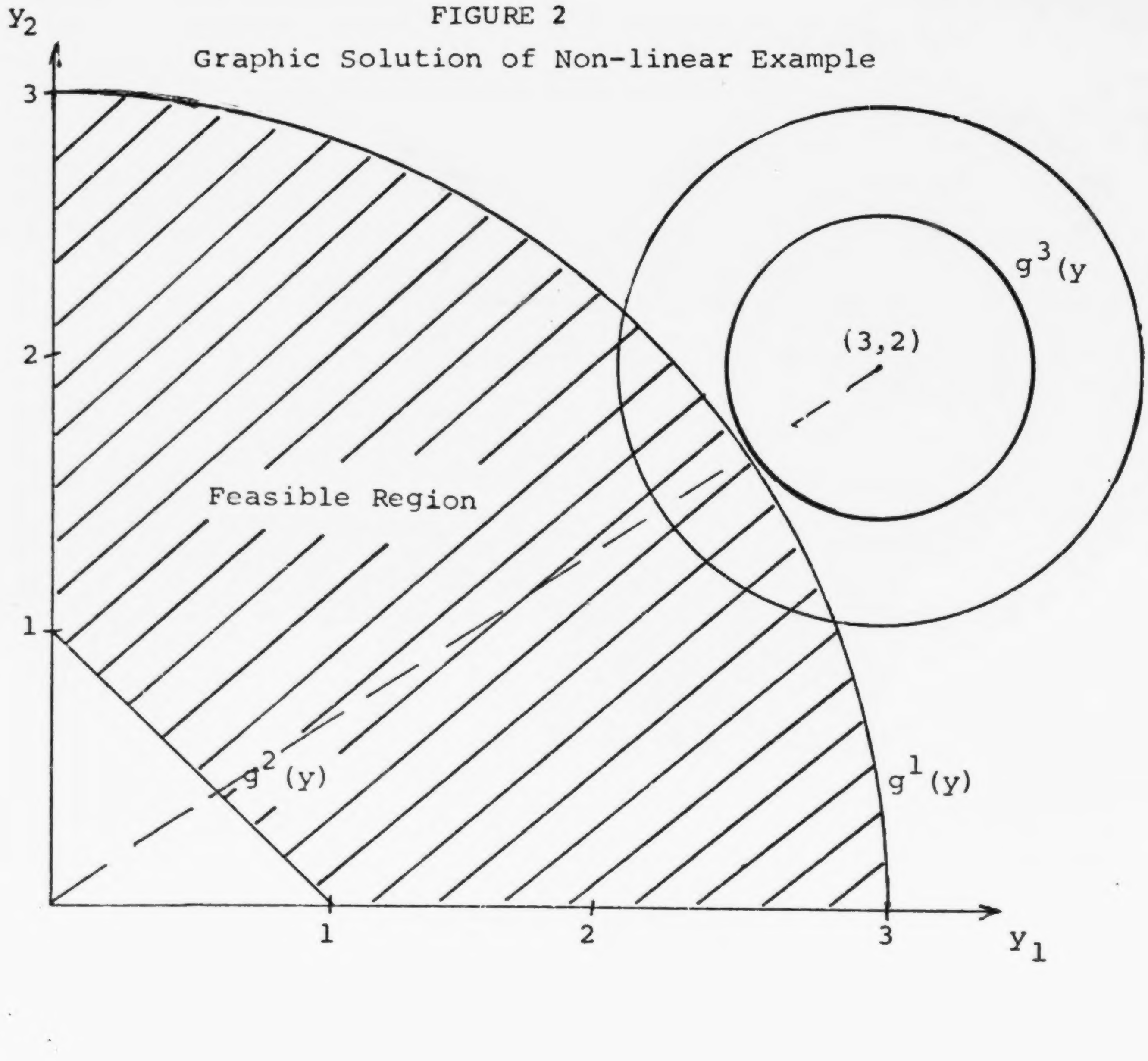
$$\min \; g^3(y) = (y_1 - 3)^2 + (y_2 - 2)^2$$

$$(y_i \geq 0)$$

Graphically the problem becomes

FIGURE 2

Graphic Solution of Non-linear Example



The optimum occurs at the intersection of the line,

$$2y_1 - 3y_2 = 0$$

connecting the centers of the circles $g^1(y)$ and $g^3(y)$ and the binding constraint circle $g^1(y) = 0$.

Eliminating y_1 from the above linear equation

$$y_1 = 3/2\ y_2$$

and substituting in $g^1(y) = 0$,

$$9/4\ y_2^2 + y_2^2 = 9$$

$$y_2 = 6/\sqrt{13}$$

plugging this value back into the expression for y_1

$$y_1 = 3/2 \cdot 6/\sqrt{13} = 9/\sqrt{13}$$

or numerically

$$y_1^* = 2.49615$$

$$y_2^* = 1.6641$$

and the optimum value is

$$g^3(y^*) = .366693$$

Now let us follow through the non-linear programming algorithm. Since we must seek the solution on a bounded domain, we superimpose the range restrictions

$$0 \leq y_1 \leq 5 \quad \text{and} \quad 0 \leq y_2 \leq 5$$

which are handled in the internal logic and not introduced explicitly. The associated local linear programming problems are:

$$\text{s.t.} \quad \nabla g^i(y^o)\cdot\Delta y \leq -g^i(y^o) - \bar{k}\, r^i$$

$$\min \quad \nabla g^m(y^o)\cdot\Delta y. \qquad \text{(see pg. 11)}$$

In our problem,

$$\nabla g^1(y) = [2y_1, 2y_2]$$

$$\nabla g^2(y) = [-1 , -1]$$

$$\nabla g^3(y) = [2(y_1 - 3) , 2(y_2 - 2)]$$

In general, r^i is taken as the error term $R^i(\Delta y)$ of the previous step and $\bar{k} = 1$. The error terms $R^i(\Delta y)$ reflect the length $|\Delta y|$ of the previous step and may cause the local linear programming problems to be inconsistent or prevent a local linear gain. However, as given on pages 17 and 18 , the $\bar{k}$ can then be adjusted down to as low as

$$\bar{k} = \epsilon/(B_3 + B_1\cdot B_3)$$

parametrically in the course of the computation to gain consistency and a local linear gain. More specifically, if an inconsistency is encountered in the local linear programming problem there will exist an index w such that

$$(-g^w(y^o)-\bar{k}r^w) - \nabla g^w(y^o)\cdot\Delta y < 0.$$

We employ (15) page 17 and calculate $\bar{k}$ to eliminate this violation or,

$$(-g^w(y^o)-\bar{k}r^w) - (\sum_{p\epsilon H} x_p(-g^p(y^o)-\bar{k}r^p)) = 0$$

$$\bar{k} = -(g^w(y^o) + \sum_{p\in H}(-x_p)\cdot g^p(y^o))/(r^w + \sum_{p\in H}(-x_p)r^p)$$

If,

$$-(g^w(y^o) + \sum_{p\in H}(-x_p)g^p(y^o)) < \epsilon$$

then re-arranging the terms,

$$\sum_{p\in H}(-x_p)\cdot g^p(y^o) > -g^w(y^o) - \epsilon$$

and we terminate with the knowledge the non-linear problem is inconsistent inasmuch as we have satisfied condition 2) page 19 of a local linear stationary point. Note if

$$-(g^w(y^o) + \sum_{p\in H}(-x_p)\cdot g^p(y^o)) \geq \epsilon$$

then by our definition of the bounds B_1, B_2 and B_3,

$$\bar{k} \geq \epsilon/(B_3 + B_1\cdot B_3)$$

and we guarantee a movement of sufficient magnitude to achieve convergence. In order to insure a local linear gain, we must have

$$\nabla g^m(y^o)\cdot \Delta y < 0$$

Using the equivalent expression given in (19) page 18 , yields

$$\sum_p(-x_p)g^p(y^o) + \bar{k}\cdot\sum_p(-x_p)r^p < 0$$

or,

$$\bar{k} < -\sum_p(-x_p)g^p(y^o)/\sum_p(-x_p)r^p$$

The dual variables x_p required for these parametric adjustments are readily available with our L.P. code.

This parametric adjustment will be illustrated in the course of solving the present problem.

Suppose we start at the point $y_o = (0,0)$ and since we have no previous step estimates, take $r^1 = r^2 = 0$. The initial linear programming problem is:

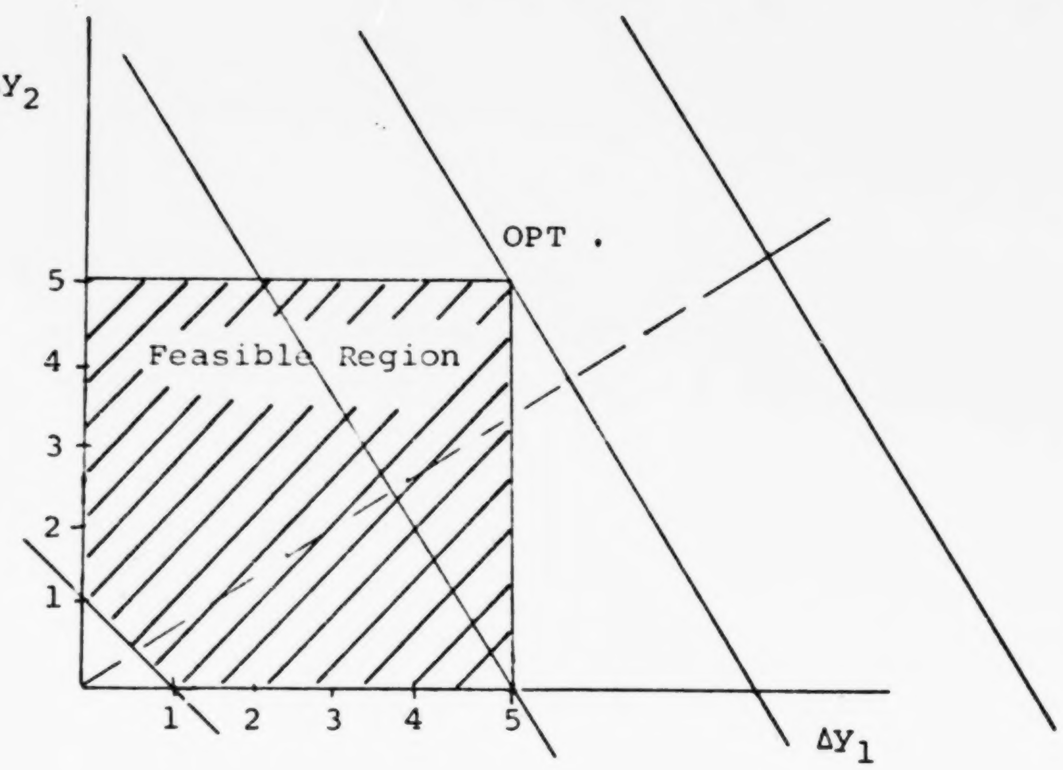
$$\begin{aligned} \text{s.t.}\quad & 0\cdot\Delta y_1 + 0\cdot\Delta y_2 \leq 9 \\ & -\Delta y_1 - \Delta y_2 \leq -1 \\ \text{min}\quad & -6\Delta y_1 - 4\Delta y_2 \end{aligned}$$

The implicit range restrictions on the variables are:

$$0 \leq \Delta y_1 \leq 5 \quad \text{and} \quad 0 \leq \Delta y_2 \leq 5$$

FIGURE 3

Graphic Solution of First L.L.P.



The optimal solution is $\Delta y = (5,5)$. The new point will be of the form,

$$y = y^o + k\Delta y = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + k \begin{bmatrix} 5 \\ 5 \end{bmatrix} .$$

where we must determine $k \leq 1$ to reduce SUPG = 1 or achieve feasibility. The error term $R^1(\Delta y)$ is obtained from

$$g^1(y) = g^1(y^o) + \nabla g^1(y^o) \cdot \Delta y + R^1(\Delta y)$$

by a single additional function evaluation of $g^1(y)$ inasmuch as $g^1(y^o)$ is already available and $\nabla g^1(y^o)$ is given by the local linear programming solution. Carrying this out,

$$R^1(\Delta y) = g^1(y) - g^1(y^o) - \nabla g^1(y^o) \cdot \Delta y$$

$$= 41 - (-9) - 0 = 50.$$

Using the quadratic approximation,

$$g^1(y) = g^1(y^o) + (\nabla g^1(y^o) \cdot \Delta y) \cdot k + R(\Delta y) \cdot k^2$$

(which happens to be exact in this case) to solve for the k yields

$$-9 + 0 \cdot k + 50 \cdot k^2 = 0$$

$$k = 3/\sqrt{50} = .424264$$

Now $R^2(\Delta y) = 0$ since the second constraint is linear and

$$g^2(y) = g^2(y^o) + \nabla g^2(y^o) \cdot \Delta y \cdot k$$

$$= 1 - 10 \cdot k = 0$$

at $k = .1$

Hence, we are feasible for any value of k in the interval $.1 \leq k \leq .424264$. We take $k = .424264$ which yields the best gain in $g^3(y)$.

The new point becomes

$$y^o = (2.121321,\ 2.121321)$$

and in setting up the next linear programming problem we have,

$$g^1(y^o) = 0. \quad , \quad g^2(y^o) = -3.242641$$

$$R^1(\Delta y^o) = 50 \quad , \quad R^2(\Delta y^o) = 0.$$

The second linear program becomes

$$\begin{array}{lllll} \text{s.t.} & & & & \\ & 4.24264\ \Delta y_1 & + & 4.24264 \cdot \Delta y_2 & \leq -50 \\ & -1.0\ \ \Delta y_1 & - & 1.0\ \ \Delta y_2 & \leq 3.24264 \\ \text{min} & -1.75736\ \Delta y_1 & + & .242641 \cdot \Delta y_2 & \end{array}$$

The range restrictions are

$$0 \leq y^o + \Delta y \leq 5$$

or

$$-2.121321 \leq \Delta y_1 \leq 2.878679$$

$$-2.121321 \leq \Delta y_2 \leq 2.878679$$

Now, if we take both Δy_1 and Δy_2 at their lower bounds, the first linearized constraint becomes,

$$4.24264 \times (-2.121321) + 4.24264 \times (-2.121321)$$

$$= -18 > -50$$

and the problem is clearly inconsistent. In the course of the solution of the linear programming problem, the initial $\overline{k} = 1$ is parametrically adjusted down to $\overline{k} = .275148$.

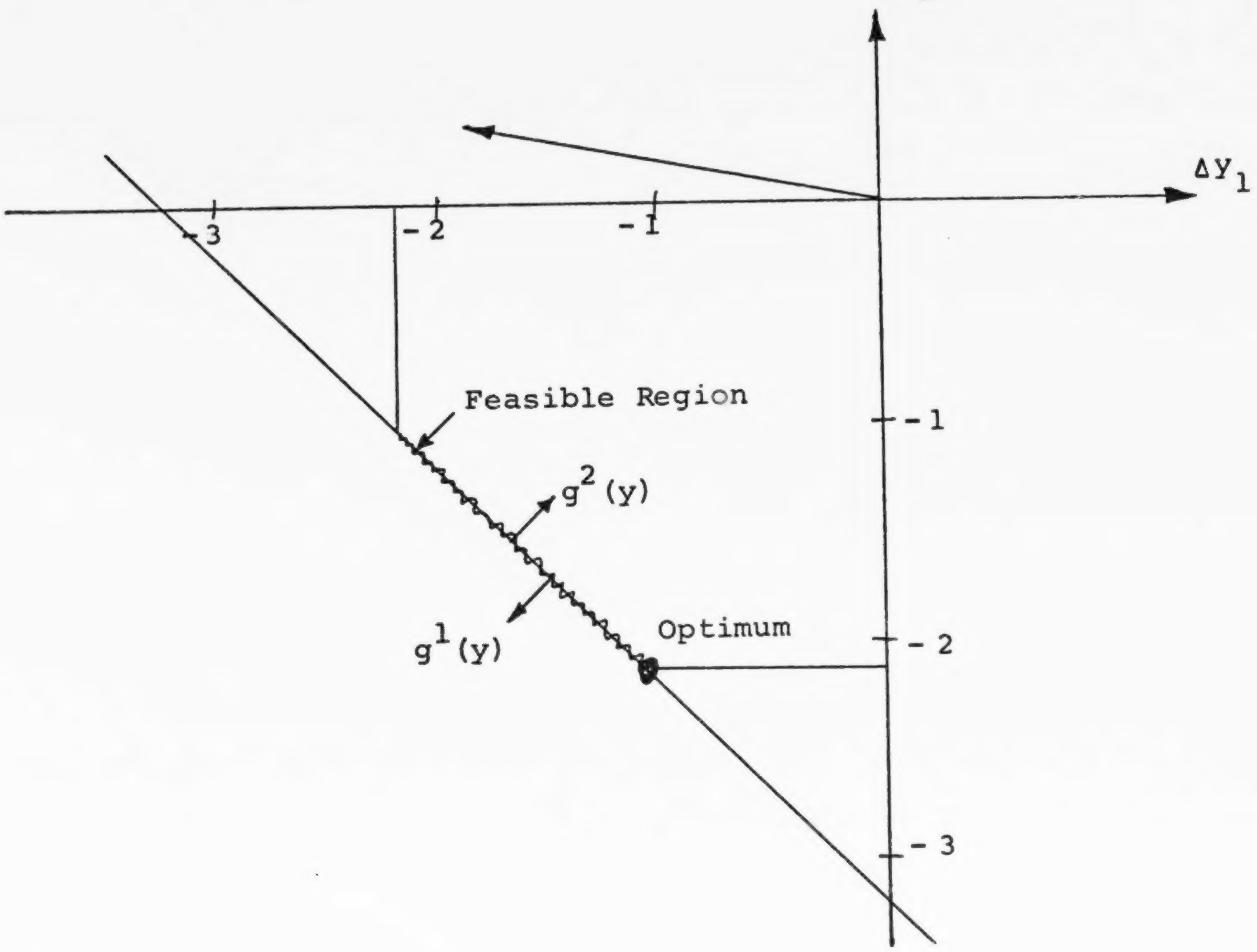
The right hand side of the adjusted linear programming problem is

$$-g^1(y^o) - \overline{k}\ R^1(\Delta y^o) = -0 - \overline{k} \cdot 50 = -13.7574$$

$$-g^2(y^o) - \overline{k}\ R^2(\Delta y^o) = -g^2(y^o) - k \cdot 0 = 3.24264$$

FIGURE 4

Graphic Solution of Second L.L.P.


Δy_2
Δy_1
-3
-2
-1
Feasible Region
$g^2(y)$
$g^1(y)$
Optimum
-1
-2
-3

The two linearized constraints were adjusted until they coincided to give a feasible region. The unrefined optimal solution is

$$\Delta y_1 = -1.12133$$

$$\Delta y_2 = -2.121321$$

The computer program at this point makes a post-optimal adjustment of $\bar{k}$ to further improve the solution of the local linear program. The optimal solution of a linear program always occurs at a basic solution. The basis consists of n linearly independent row vectors or constraints where for theoretical purposes we include the range constraints of the form

$$\underline{e}_j \ \Delta y \leq U_j - y_j^o$$

$$-\underline{e}_j \ \Delta y \leq y_j^o - L_j$$

at the present optimal solution the basis is:

$$B = \begin{bmatrix} 4.24264 & 4.24264 \\ 0 & -1 \end{bmatrix}$$

This consists of the first constraint and the lower bound constraint on Δy_2. The optimal solution is

$$\Delta y = B^{-1} \ (-g - \bar{k}r).$$

In this particular instance,

$$B^{-1} = \begin{bmatrix} .235702 & 1 \\ 0 & -1 \end{bmatrix}$$

and,

$$\Delta y = \begin{bmatrix} .235702 & 1 \\ 0 & -1 \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ 2.121321 \end{bmatrix} + \bar{k} \begin{bmatrix} -50 \\ 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 2.121321 \\ -2.121321 \end{bmatrix} + \bar{k} \begin{bmatrix} -11.7851 \\ 0 \end{bmatrix}.$$

With our current adjusted $\bar{k} = .275148$ we obtain,

$$\Delta y = \begin{bmatrix} -1.12133 \\ -2.121321 \end{bmatrix}$$

Now observe,

$$\nabla g^3(y^o)\cdot\Delta y = (-1.75736, .242641) \left\{ \begin{bmatrix} 2.121321 \\ -2.121321 \end{bmatrix} + \bar{k} \begin{bmatrix} -11.7851 \\ 0 \end{bmatrix} \right\}$$

$$= -4.24264 + \bar{k}\cdot 20.71066$$

With our current adjusted $\bar{k} = .275148$

$$\nabla g^3(y^o)\cdot\Delta y = 1.455856,$$

and since this is positive, we can not even obtain or achieve a local gain. In general, as $\bar{k} \rightarrow 0$, we obtain the best "rate of gain", but the distance we can move approaches zero when we are on a boundary.

Distance Moved

Suppose one of the constraining functions $g^i(y^o) = 0$ (on boundary) and we return to this boundary after movement,

1) $g^i(y) = \nabla g^i(y^o)\cdot\Delta y\cdot k + R^i(k\Delta y) = 0$

From the local linear program the linear gain is,

2) $\nabla g^i(y^o)\cdot\Delta y = -\bar{k}\ r^i$

Substituting expression 2) into expression 1) and using the approximation: $R^i(k\Delta y) = k^2\cdot R^i(\Delta y)$ yields,

$$k\cdot(-\bar{k}\cdot r^i + k\cdot R^i(\Delta y)) = 0$$

and we are at the boundary for $k = 0$ (no movement) and for,

$$k = \bar{k}\cdot\frac{r^i}{R^i(\Delta y)}$$

Now we neglect changes in the error term due to direction (with circles there is no error since they have uniform curvature). Since we always take $r^i = R^i(\Delta y^o)$ this assumption translates to

$$\frac{r^i}{R^i(\Delta y)} = \frac{R^i(\Delta y^o)}{R^i(\Delta y)} = \frac{\Delta y^{o^T}\Delta y^o}{\Delta y^T\Delta y}$$

and the distance moved becomes,

$$k = \bar{k}\ \frac{(\Delta y^{o^T}\Delta y^o)}{(\Delta y^T\Delta y)}$$

Assuming we move this distance, the objective is to minimize

$$g^m(y) = g^m(y^o) + \nabla g^m(y^o)\cdot\Delta y\cdot k + R(\Delta y)\cdot k^2$$

and employing our assumption that

$$R^i(\Delta y) = \frac{\Delta y^T \Delta y}{\Delta y^{o^T} \Delta y^o} \cdot R^i(\Delta y^o)$$

as well as our expression for k yields

$$g^m(y) = g^m(y^o) + \nabla g^m(y^o) \cdot \Delta y \cdot \frac{\bar{k}(\Delta y^{o^T} \Delta y^o)}{\Delta y^T \Delta y}$$

$$+ \frac{\bar{k}^2 (\Delta y^{o^T} \Delta y^o)^2}{(\Delta y^T \Delta y)^2} \cdot \frac{\Delta y^T \Delta y}{\Delta y^{o^T} \Delta y^o} \cdot R^m(\Delta y^o)$$

$$= g^m(y^o) + \frac{(\nabla g^m(y^o) \cdot \Delta y \cdot \bar{k} + \overset{m}{R}(\Delta y^o) \cdot \bar{k}^2) \cdot \Delta y^{o^T} \Delta y^o}{\Delta y^T \Delta y}$$

Now defining

$$\underline{d} = B^{-1} g \text{ and } \underline{p} = B^{-1} r^i$$

we have

$$\Delta y = \underline{d} + \bar{k}\, \underline{p}$$

and

$$\Delta y^T \Delta y = \underline{d}^T \underline{d} + 2\, \underline{d}^T \underline{p} \cdot \bar{k} + (\underline{p}^T \cdot p) \cdot \bar{k}^2$$

Further defining $dm = \nabla g^m(y^o) \cdot \underline{d}$, $pm = \nabla g^m(y^o) \cdot \underline{p}$

$$\nabla g^m(y^o) \cdot \Delta y = \nabla g^m(y^o) \cdot \underline{d} + \bar{k} \cdot \nabla g^m(y^o) \cdot \underline{p} = dm + \bar{k} pm$$

With the additional definition of

$$p\,m\,r = p\,m + R^m(\Delta y^o)$$

we can rewrite the expression for $g^m(y)$ as:

$$g^m(y) = g^m(y^o) + \frac{(dm\bar{k} + pmr\bar{k}^2) \cdot (\Delta y^{o^T} \Delta y^o)}{\underline{d}^T d + 2\underline{d}^T p \cdot \bar{k} + \underline{p}^T \underline{p} \bar{k}^2}$$

To obtain the minimum value of $g^m(y)$, we set the derivative

$$\frac{d\, g^m(y^o)}{d\, \overline{K}} = 0$$

and solve for $\overline{k}$.

Working through the algebra

$$\overline{k}^* = \frac{-B + \sqrt{B^2 - AC}}{A}$$

where,

$$A = 2 \cdot (\underline{d}^T \underline{p}) \cdot pmr - dm \cdot (\underline{p}^T p)$$
$$B = (\underline{d}^T \underline{d}) \cdot pmr$$
$$C = (\underline{d}^T \underline{d}) \cdot dm$$

In our example,

$(\underline{p}^T \underline{p}) = 138.88882$

$(\underline{d}^T \underline{p}) = -25.$

$(\underline{d}^T \underline{d}) = 9.$

$pmr = 70.710663$

$dm = -4.24263$

$A = -2946.286$

$B = 636.39746$

$C = -38.183762$

$\overline{k}^* = .0324352$

Using this optimal $\bar{k}^*$ yields

$$\Delta Y = \begin{bmatrix} 2.121321 \\ \\ -2.121321 \end{bmatrix} + .0324352 \begin{bmatrix} -11.7851 \\ \\ 0 \end{bmatrix} = \begin{bmatrix} 1.739072 \\ \\ -2.121321 \end{bmatrix}$$

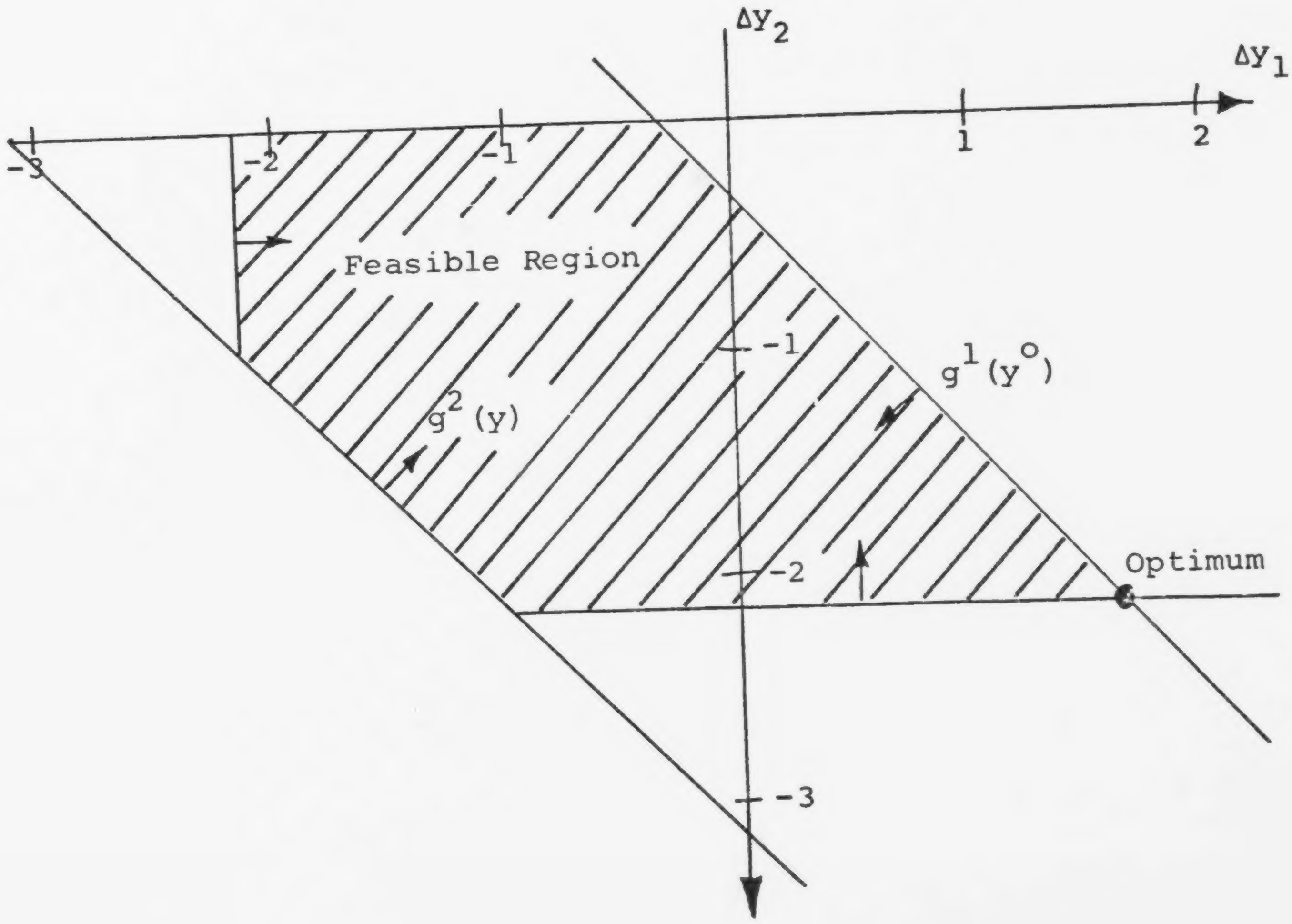
and the adjusted right hand side of the underlying linear program becomes

$$-g^1(y^o) - \bar{k}^* R^1(\Delta y^o) = 0 - \bar{k} \cdot 50 = -1.62175$$

$$-g^2(y^o) - \bar{k}^* R^2(\Delta y^o) = g^2(y^o) - \bar{k} \cdot 0 = 3.24264$$

FIGURE 5

Graphic Solution of Second L.L.P.
After Parametric Adjustment



Again using the formula,

$$R^1(\Delta y) = g^1(y) - g(y^o) - \nabla g^1(y^o)\cdot\Delta y$$

we calculate

$$R^1(\Delta y) = 7.524391$$

Using the quadratic approximation,

$$g^1(y) = g^1(y^o)+(\nabla g^1(y^o)\cdot\Delta y)\cdot k+R(\Delta y)\cdot k^2$$

(which is exact in this case) to solve for k yields

$$0 - 1.621761\cdot k + 7.524391\, k^2 = 0$$

$$k = \frac{1.621761}{7.524391} = .2155338$$

Now $R^2(\Delta y) = 0$ since the second constraint is linear and

$$g^2(y) = g^2(y^o) + \nabla g^2(y^o)\cdot\Delta y\cdot k$$

$$= -3.242641 + .3822479\cdot k = 0$$

for $k \leq 8.483084$.

Hence, we are feasible for

$$0 \leq k \leq .2155338$$

and the optimum value of $g^3(y)$ on this interval occurs for $k = .2155338$.

Using this value of k, we calculate

$$y = y_o + k\cdot\Delta y = \begin{bmatrix} 2.121321 \\ 2.121321 \end{bmatrix} + .215534 \begin{bmatrix} 1.739072 \\ -2.121321 \end{bmatrix}$$

$$= \begin{bmatrix} 2.496150 \\ 1.664104 \end{bmatrix}$$

the optimum solution. The optimum value is

$$g^3(y) = .3666912.$$

It should be observed that in making the parametric adjustments to $\overline{k}$, we may transfer basis in the linear programming tableau. This requires calculating the range of adjustment of $\overline{k}$ to preserve the current basis and making additional pivots if we adjust beyond these limits.

III. DERIVATION OF THE NON-LINEAR PROBLEM

In this section, we will derive the constraints and objective function of a comprehensive non-linear programming problem for the regional management of water pollution control. In order to facilitate the derivation we will set up a small scale problem. In Figure 6 we have divided a length of estuary into three sections numbered 1, 2, 3. We have located five polluters numbered 1,2,3,4,5 along the length of the estuary discharging waste water into the section denoted by the arrow. Finally we have fixed the location of three regional treatment plants numbered 1, 2, 3. The regional plants can be located from political or engineering considerations. Note that to consider the location of the regional treatment plants as variable introduces a combinatorial programming problem mapping polluters into regional plants and regional plants into sections. This combinatorial programming problem will be dealt with in subsequent work. In this study, we take the plant locations as fixed. Finally, observe that if no waste water is piped to a particular regional plant, then that regional plant is not opened.

Suppose that we have obtained from each polluter the current flow $\bar{f}_{ij}$ in (MGD), where $\bar{f}_{ij}$ is the flow from polluter j to section i, and the current BOD concentration $\bar{j}_i$ in lb/MG of effluent discharged to the estuary (see Table 1).

FIGURE 6

Physical Layout for Example

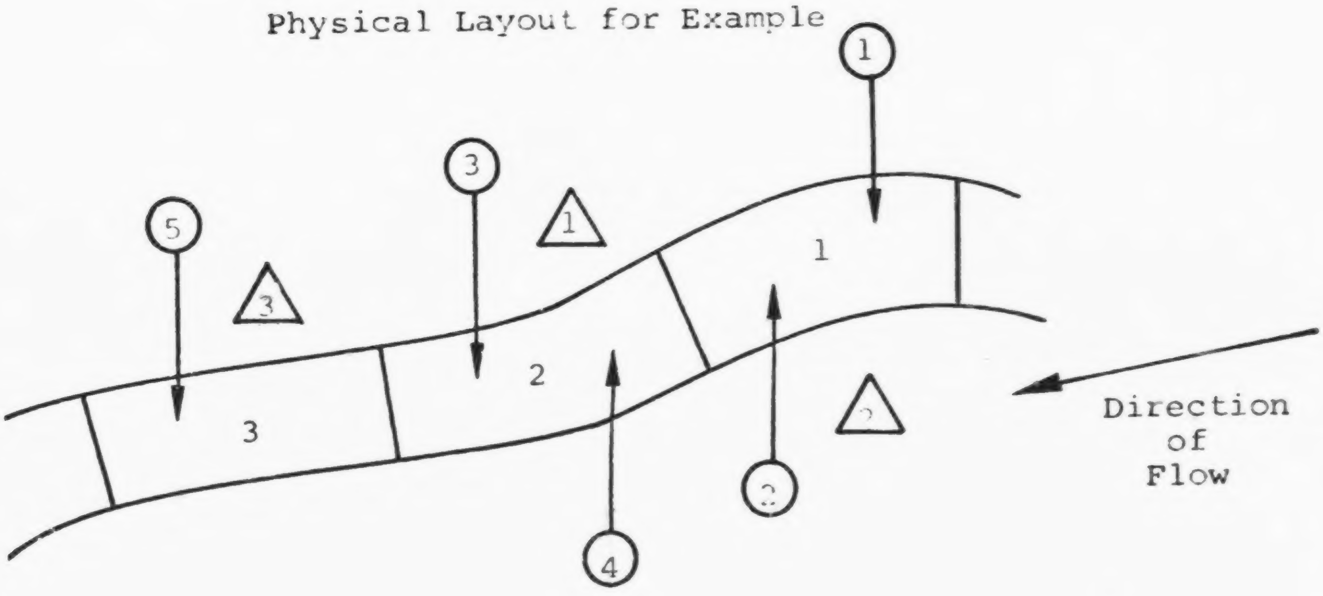


TABLE 1

Flows and Concentrations for Example Estuary

Polluter Number	Current Flow $\bar{f}_{ij}$ (MGD)	Current BOD $\bar{j}_i$ (lb/MG)
1	19.7	155.
2	7.0	1801.
3	3.0	666.
4	6.1	278.
5	4.0	557.

In order to compute our piping costs we have obtained the following distances all measured in miles: (1) d^1_{ij}, the distance from polluter j to the center of section i (See Table 2); (2) d^2_{ij}, the distance from polluter j to regional treatment plant i with ∞ indicating a very large number to prohibit piping wastes across the estuary (See Table 3); and (3) d_{ij}^{3}, the distance from regional treatment plant j to section i (See Table 4).

TABLE 2

Distance Polluter to Section for Example Estuary

		Polluter				
		1	2	3	4	5
Section	1	0	0	2	2	6
	2	3	2	0	0	3
	3	5	5	3	3	0

TABLE 3

Distance Polluter to Plant for Example Estuary

		Polluter				
		1	2	3	4	5
Regional Plant	1	2	∞	1	∞	4
	2	∞	1	∞	2	∞
	3	4	∞	3	∞	1

TABLE 4

Distance Plant to Section for Example Estuary

		Regional Plant		
		1	2	3
Section	1	1	0.1	5
	2	0.1	2.	2
	3	4.	6.	0.1

Lastly we have a 3 x 3 steady state response matrix of transfer coefficients a_{ij} in mg/l per pound per day (mg/l/lb/day) (See Table 5). The transfer coefficient is defined as the change in DO in section i of the estuary due to a change in input of BOD in section j of the estuary. Thus, in Table 5 an input into section 1 of 100,000 lbs/day will produce a DO decrease in section 3 of 0.8421 mg/l.

TABLE 5

Transfer Coefficients for Example Estuary

		Section Number		
		1	2	3
Section Number	1	1.096×10^{-5}	5.328×10^{-6}	2.214×10^{-6}
	2	1.047×10^{-5}	9.817×10^{-6}	4.854×10^{-6}
	3	8.421×10^{-6}	9.431×10^{-6}	9.108×10^{-6}

The five waste discharges into the estuary will produce a DO profile down the length of the estuary for steady state conditions. Now suppose some regulatory agency has decided to change this profile and has given us a vector of DO changes, c_i, in (mg/l) (See Table 6). Thus our problem can be stated as follows: we must achieve the desired DO changes in each section at a minimum overall cost.

TABLE 6

DO Changes for Example Estuary

		DO Change Vector mg/l
Section Number	1	+ 0.12
	2	0.00
	3	- 0.12

In deriving the mathematical programming formulation, we will need the following additional notation:

f_{ij} the flow from polluter j to section i in (MGD)

p_{ij} the flow from polluter j to treatment plant i in (MGD)

t_{ij} the flow from treatment plant j to section i in (MGD)

j_i the BOD concentration of effluent from polluter i (lb/MG)

r_i the fractional removal of BOD at treatment plant i

The corresponding matrices (F, P, T, J, and R) of these five quantities completely specify the variables of the problem. Finally we will adopt the convention that a variable with a bar, such as $\bar{f}_{ij}$, denotes the current value of this variable. The mathematical programming formulation of this problem has three sets of constraints.

Set 1: Material Balance at a Polluter

In matrix notation we have

$$F^T \cdot 1 + P^T \cdot 1 = \bar{F}^T \cdot 1 + \bar{P}^T \cdot 1$$

which expands to

$$\begin{bmatrix} f_{11} & f_{21} & f_{31} \\ f_{12} & f_{22} & f_{32} \\ f_{13} & f_{23} & f_{33} \\ f_{14} & f_{24} & f_{34} \\ f_{15} & f_{25} & f_{35} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} p_{11} & p_{21} & p_{31} \\ p_{12} & p_{22} & p_{32} \\ p_{13} & p_{23} & p_{33} \\ p_{14} & p_{24} & p_{34} \\ p_{15} & p_{25} & p_{35} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \bar{f}_{11} & 0 & 0 \\ \bar{f}_{12} & 0 & 0 \\ 0 & \bar{f}_{23} & 0 \\ 0 & \bar{f}_{24} & 0 \\ 0 & 0 & \bar{f}_{35} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \odot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus we see that the matrix multiplications give the desired set of constraints.

Set 2: Material Balance at a Regional Plant

In matrix notation we have

$$P \cdot 1 = T^T \cdot 1$$

which expands to

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{21} & t_{31} \\ t_{12} & t_{22} & t_{32} \\ t_{13} & t_{23} & t_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Set 3: Quality Constraints

The third set of constraints, called the quality constraints, is more complicated. The quality constraint for section 1 is:

$$a_{11}\begin{bmatrix}\text{Change in BOD} \\ \text{input to section 1}\end{bmatrix} + a_{12}\begin{bmatrix}\text{Change in BOD} \\ \text{input to section 2}\end{bmatrix} + a_{13}\begin{bmatrix}\text{Change in BOD} \\ \text{input to section 3}\end{bmatrix} \geq c_1$$

From the first term we need the change in BOD input to section 1 in lbs/day. There are two components of this change: (1) the direct discharge (polluter to section), and (2) the indirect discharge (treatment plant to section). For the first term the direct effect is

$$a_{11}(\bar{f}_{11}\bar{j}_1+\bar{f}_{12}\bar{j}_2-f_{11}j_1-f_{12}j_2-f_{13}j_3-f_{14}j_4-f_{15}j_5).$$

The indirect effect is:

$$a_{11}\Big[(1-\bar{r}_1)(\bar{p}_{11}\bar{j}_1 + \bar{p}_{12}\bar{j}_2 + \bar{p}_{13}\bar{j}_3 + \bar{p}_{14}\bar{j}_4 + \bar{p}_{15}\bar{j}_5)\left(\frac{\bar{t}_{11}}{\bar{t}_{11}+\bar{t}_{21}+\bar{t}_{31}}\right)$$

$$-(1-r_1)(p_{11}j_1 + p_{12}j_2 + p_{13}j_3 + p_{14}j_4 + p_{15}j_5)\left(\frac{t_{11}}{t_{11}+t_{21}+t_{31}}\right)$$

$$+(1-\bar{r}_2)(\bar{p}_{21}\bar{j}_1 + \bar{p}_{22}\bar{j}_2 + \bar{p}_{23}\bar{j}_3 + \bar{p}_{24}\bar{j}_4 + \bar{p}_{25}\bar{j}_5)\left(\frac{\bar{t}_{12}}{\bar{t}_{12}+\bar{t}_{22}+\bar{t}_{32}}\right)$$

$$-(1-r_2)(p_{21}j_1 + p_{22}j_2 + p_{23}j_3 + p_{24}j_4 + p_{25}j_5)\left(\frac{t_{12}}{t_{12}+t_{22}+t_{32}}\right)$$

$$+(1-\bar{r}_3)(\bar{p}_{31}\bar{j}_1 + \bar{p}_{32}\bar{j}_2 + \bar{p}_{33}\bar{j}_3 + \bar{p}_{34}\bar{j}_4 + \bar{p}_{35}\bar{j}_5)\left(\frac{\bar{t}_{13}}{\bar{t}_{13}+\bar{t}_{23}+\bar{t}_{33}}\right)$$

$$-(1-r_3)(p_{31}j_1 + p_{32}j_2 + p_{33}j_3 + p_{34}j_4 + p_{35}j_5)\left(\frac{t_{13}}{t_{13}+t_{23}+t_{33}}\right)\Big].$$

We can combine the direct and indirect effects and simplify to give:

$$a_{11}\Big[\bar{f}_{11}\bar{j}_1+\bar{f}_{12}\bar{j}_2-f_{1.}J+\frac{(1-\bar{r}_1)(\bar{p}_{1.}\bar{J})\bar{t}_{11}}{1^T\bar{t}_{.1}}-\frac{(1-r_1)(p_{1.}J)t_{11}}{1^Tt_{.1}}$$

$$+\frac{(1-\bar{r}_2)(\bar{p}_{2.}\bar{J})\bar{t}_{12}}{1^T\bar{t}_{.2}}-\frac{(1-r_2)(p_{2.}J)t_{12}}{1^Tt_{.2}}$$

$$+\frac{(1-\bar{r}_3)(\bar{p}_{3.}\bar{J})\bar{t}_{13}}{1^T\bar{t}_{.3}}-\frac{(1-r_3)(p_{3.}J)t_{13}}{1^Tt_{.3}}\Big].$$

The second and third terms of the quality constraint are

$$a_{12}\Big[\bar{f}_{23}\bar{J}_3 + \bar{f}_{24}\bar{J}_4 - f_2.J + \frac{(1-\bar{r}_1)(\bar{p}_1.\bar{J})\bar{t}_{21}}{1^T\bar{t}_{\cdot 1}} - \frac{(1-r_1)(p_1.J)t_{21}}{1^Tt_{\cdot 1}}$$

$$+ \frac{(1-\bar{r}_2)(\bar{p}_2.\bar{J})\bar{t}_{22}}{1^T\bar{t}_{\cdot 2}} - \frac{(1-r_2)(p_2.J)t_{22}}{1^Tt_{\cdot 2}}$$

$$+ \frac{(1-\bar{r}_3)(\bar{p}_3.\bar{J})\bar{t}_{23}}{1^T\bar{t}_{\cdot 3}} - \frac{(1-r_3)(p_3.J)t_{23}}{1^Tt_{\cdot 3}}\Big] +$$

$$a_{13}\Big[\bar{f}_{35}\bar{J}_5 - f_3.J + \frac{(1-\bar{r}_1)(\bar{p}_1.\bar{J})\bar{t}_{31}}{1^T\bar{t}_{\cdot 1}} - \frac{(1-r_1)(p_1.J)t_{31}}{1^Tt_{\cdot 1}}$$

$$+ \frac{(1-\bar{r}_2)(\bar{p}_2.\bar{J})\bar{t}_{32}}{1^T\bar{t}_{\cdot 2}} - \frac{(1-r_2)(p_2.J)t_{32}}{1^Tt_{\cdot 2}}$$

$$+ \frac{(1-\bar{r}_3)(\bar{p}_3.\bar{J})\bar{t}_{33}}{1^T\bar{t}_{\cdot 3}} - \frac{(1-r_3)(p_3.J)t_{33}}{1^Tt_{\cdot 3}}\Big].$$

Now combine the three terms and multiply through by (-1) to give:

$$A_1.FJ + A_1.\Big[\frac{(1-r_1)(p_1.J)t_{\cdot 1}}{1^Tt_{\cdot 1}} + \frac{(1-r_2)(p_2.J)t_{\cdot 2}}{1^Tt_{\cdot 2}} + \frac{(1-r_3)(p_3.J)t_{\cdot 3}}{1^Tt_{\cdot 3}}\Big]$$

$$+ c_1 - A_1.\bar{F}\bar{J} - A_1.\Big[\frac{(1-\bar{r}_1)(\bar{p}_1.\bar{J})\bar{t}_{\cdot 1}}{1^T\bar{t}_{\cdot 1}} + \frac{(1-\bar{r}_2)(\bar{p}_2.\bar{J})\bar{t}_{\cdot 2}}{1^T\bar{t}_{\cdot 2}} +$$

$$\frac{(1-\bar{r}_3)(\bar{p}_3.\bar{J})\bar{t}_{\cdot 3}}{1^T\bar{t}_{\cdot 3}}\Big] \leq 0.$$

Generally we will have k treatment plants which gives the generalized expression:

$$A_1.FJ + A_1. \sum_{i=1}^{k} \frac{t_{.i}(1-r_i)(p_i.J)}{1^T t_{.i}} + c_1$$

$$- A_1.\bar{F}\bar{J} - A_1. \sum_{i=1}^{k} \frac{\bar{t}_{.i}(1-\bar{r}_i)(\bar{p}_i.\bar{J})}{1^T \bar{t}_{.i}} \leq 0.$$

Now we generalize for all three sections to obtain the desired result:

$$AFJ + A \sum_{i=1}^{k} \frac{t_{.i}(1-r_i)(p_i.J)}{1^T t_{.i}} + C - A\bar{F}\bar{J} - A \sum_{i=1}^{k} \frac{\bar{t}_{.i}(1-\bar{r}_i)(\bar{p}_i.\bar{J})}{1^T \bar{t}_{.i}} \leq 0.$$

The two sets of material balance equations and the set of quality constraints constitute the explicit constraints of this problem. For this problem, the three sets of explicit constraints and the five blocks of variables can be conveniently represented in a schematic as follows.

Constraints \ Variables		15	30	39	44	47
Material Balance at a Polluter	5	f_{ij}	p_{ij}	⊙	⊙	⊙
Material Balance at a Regional Plant	8	⊙	p_{ij}	t_{ij}	⊙	⊙
Quality Constraints	11	f_{ij}	p_{ij}	t_{ij}	j_i	r_i

Thus the local linear programming problem has 11 constraints and 47 variables in addition to the upper and lower bound constraints. By eliminating certain blocks of variables (and, or blocks of constraints), we can consider a number of different problem combinations. These combinations are:

(1) treatment at polluter only;

(2) by-pass piping only;

(3) treatment at regional plants only;

(4) treatment at polluter and by-pass piping;

(5) treatment at polluter and treatment at regional plants;

(6) by-pass piping and treatment at regional plants; and

(7) by-pass piping, treatment at polluter, and treatment at regional plants.

The combination we are specifically interested in is number 7. Combinations 1 and 2 can be reduced to linear programming problems and have already been solved. The linear programming problem for treatment at the polluter (case 1) has been solved by Thomann and Marks [8]. Their formulation can be obtained from the overall problem very easily. Note from the schematic that the variable j_i appears only in the quality constraints. The general quality constraints are:

$$AFJ + A\sum_{i=1}^{k} \frac{t_{\cdot i}(1-r_i)(p_{i\cdot}J)}{1^T t_{\cdot i}} + C - A\bar{F}\bar{J} - A\sum_{i=1}^{k} \frac{\bar{t}_{\cdot i}(1-\bar{r}_i)(\bar{p}_{i\cdot}\bar{J})}{1^T \bar{t}_{\cdot i}} \leq 0$$

which reduce to

$$A\bar{F}\Delta J \geq C \quad \text{(where } \Delta J = \bar{J} - J\text{)}$$

inasmuch as $P = \odot$, $T = \odot$, and $F = \bar{F}$ without treatment plants or by-pass piping. Recall that j_i, the current concentration, is used to reflect additional treatment at the polluter. Imposing a policy that polluters shall not decrease their level of treatment (or treat negative amounts) leads to the range restrictions:

$$0 \leq j_i \leq \bar{U}_i/\bar{f}_i \text{ where } j_i = U_i/\bar{f}_i$$

The objective function is

$$\sum_i s_i \Delta U_i = \sum_i s_i \ (\bar{j}_i\bar{f}_i - j_i\bar{f}_i) = \sum_i (s_i\bar{f}_i)\cdot\Delta j_i$$

where s_i (\$/lb/day) is the cost of removing an additional pound per day. The possibility of multiple price breaks is treated in detail in Section V.

The linear programming problem for by-pass piping (case 2) has been solved by Graves, Hatfield, and Whinston [7]. For this formulation the variables are the f_{ij} and the j_i are fixed at their current values $\bar{j}_i$. The first set of constraints in the overall problem (material balance at a polluter) reduce to

$$F^T\cdot 1 = \bar{F}^T\cdot 1.$$

There are no material balance at a regional plant constraints and the quality constraints reduce to

$$A\bar{F}\bar{J} - AFJ \geq C.$$

Let $J = \bar{J}$ which gives $A(\bar{F} - F)\bar{J} \geq C$. Multiply through by (-1) and re-arrange to give

$$AF\bar{J} \leq -C + A\bar{F}\bar{J}.$$

The objective function is given by

$$\sum_{ij} c_{ij} f_{ij}$$

where the piping cost coefficient c_{ij} is given by ($c_{ij}=1865 \cdot d_{ij}^{1} \cdot Q_{ij}^{-.402}$).

To complete this section, we introduce the concept of a priority class for the variables. Basically, the idea is to solve the problem with only a small subset of the variables active (priority class 1) and the rest set to zero. The problem is then resolved with the variables in priority classes 1 and 2 active and the remainder of the variables set to zero. Note that we can have as many priority classes as we want. By putting the more important variables in lower priority classes, computational efficiency is gained as we step up through the priority classes. This is because the values of the critical variables will tend to stabilize and the number of pivots to achieve feasibility and optimality will be reduced. As an example, we might want to arrange the variables in the following five priority classes.

TABLE 7

Example of Priority Class

1	$\{j_1, j_2, j_3, j_4, j_5, f_{11}, f_{12}, f_{23}, f_{24}, f_{35}\}$
2	$\{p_{11}, p_{22}, p_{13}, p_{24}, p_{35}, t_{21}, t_{12}, t_{33}\}$
3	$\{p_{12}, p_{14}, p_{15}, p_{21}, p_{23}, p_{25}, p_{31}, p_{32}, p_{33}, p_{34}, t_{11}, t_{13}, t_{22}, t_{23}, t_{31}, t_{32}\}$
4	$\{f_{22}, f_{13}, f_{14}, f_{15}, f_{21}, f_{34}, f_{25}, f_{31}, f_{32}, f_{33}\}$
5	$\{r_1, r_2, r_3\}$

Note that the solution to priority class 1 gives us the solution to case 1 (treatment at the polluter only). Opening up priority class 2 allows each polluter to pipe to the nearest regional treatment plant and each regional plant discharges to the nearest section. Priority class 3 opens up all other combinations of polluter to regional plant and regional plant to section. Priority class 4 opens up all by-pass pipes and priority class 5 allows for variable treatment efficiencies at the regional plants.

The development of the costs of treatment at the polluters and the regional plants is given in Section V. In Section VI, we present the solution to the small scale problem of this chapter for a number of different problem combinations.

IV. GENERATION OF THE UNDERLYING ARRAY OF THE LOCAL L.P.

In Section II we reviewed the truncated tableau and generation of elements technique for solving linear programming problems. We required the underlying array to be available from secondary storage or by generation. The underlying array of the local L.P. can be conveniently represented by a schematic as follows:

Material Balance at a Polluter	$\frac{\partial G^1}{\partial f_{ij}}$	$\frac{\partial G^1}{\partial p_{ij}}$	$\odot$	$\odot$	$\odot$
Material Balance at a Regional Plant	$\odot$	$\frac{\partial G^2}{\partial p_{ij}}$	$\frac{\partial G^2}{\partial t_{ij}}$	$\odot$	$\odot$
Quality Constraints	$\frac{\partial G^3}{\partial f_{ij}}$	$\frac{\partial G^3}{\partial p_{ij}}$	$\frac{\partial G^3}{\partial t_{ij}}$	$\frac{\partial G^3}{\partial j_i}$	$\frac{\partial G^3}{\partial r_i}$

Note that each column of this array corresponds to a particular variable. Thus, given any variable, the problem is to generate the corresponding column of partial derivatives. Mathematically, we need the matrix derivatives with respect to the particular variable. Systematic procedures for obtaining matrix derivatives are given by Dwyer and Macphail [9].

At this point we present the matrix derivatives that we will need in the development that follows. Let Y be a matrix whose elements are functions of the elements $x_{m,n}$ of X and let A, B, and C be matrices whose elements are not functions of $x_{m,n}$. Define $K_{m,n}$ as the matrix having the dimensions of X with all elements zero except for a unit element in the mth row and the nth column. The formulas for the derivatives are given in the following table.

TABLE 8

Matrix Derivatives

	FORM	$\frac{\partial Y}{\partial x_{m,n}}$
(1)	$Y = XB$	$K_{m,n}B$
(2)	$Y = X^TB$	$K^T_{n,m}B$
(3)	$Y = AX$	$AK_{m,n}$
(4)	$Y = AXC$	$AK_{m,n}C$

The material balance at a polluter constraints are $G^1 = F^T \cdot 1 + P^T \cdot 1 - \bar{F}^T \cdot 1 - \bar{P}^T \cdot 1 = 0$.

The $\frac{\partial G^1}{\partial f_{m,n}}$ has the form (2) and we write $\frac{\partial G^1}{\partial f_{m,n}} = K^T_{n,m} \cdot 1$ where $K_{m,n}$ has the dimension of F. Similarly the $\frac{\partial G^1}{\partial P_{m,n}}$ has the form (2) and we write $\frac{\partial G^1}{\partial P_{m,n}} = K^T_{n,m} \cdot 1$ where $K_{m,n}$ has the dimension of P.

The material balance at a regional plant type constraints are

$$G^2 = P\cdot 1 - T^T\cdot 1 = 0.$$

The $\partial G^2/\partial p_{m,n}$ has the form (1) and we write $\partial G^2/\partial p_{m,n} = K_{m,n}\cdot 1$ where $K_{m,n}$ has the dimension of P. Similarly, the $\partial G^2/\partial t_{m,n}$ has the form (2) and we write $\partial G^2/\partial t_{m,n} = -K^T_{n,m}\cdot 1$ where $K_{m,n}$ has the dimension of T. The quality constraints can be written

$$G^3 = AFJ + A\sum_{i=1}^{k}\frac{t_{\cdot i}(1-r_i)(p_{i\cdot}J)}{1^T t_{\cdot i}} +$$

$$C - A\bar{F}\bar{J} - A\sum_{i=1}^{k}\frac{\bar{t}_{\cdot i}(1-\bar{r}_i)(\bar{p}_{i\cdot}\bar{J})}{1^T\bar{t}_{\cdot i}} \le 0.$$

Block 1:

The $\partial G^3/\partial f_{m,n}$ has the form (4) and we write $\partial G^3/\partial f_{m,n} = AK_{m,n}J$ where $K_{m,n}$ has the dimension of F.

Block 2:

The $\partial G^3/\partial p_{m,n}$ has the form (4) which gives

$$\frac{\partial G^3}{\partial p_{m,n}} = \frac{(1-r_m)}{1^T t_{\cdot m}}\left[At_{\cdot m}(K_{m\cdot}J)\right]$$

where $K_{m\cdot}$ is a row vector with a 1 in the n^{th} column. The row has the same dimension as $p_{m\cdot}$.

Block 3:

For $\partial G^3/\partial t_{m,n}$ we have

$$\frac{\partial G^3}{\partial t_{m,n}} = \frac{(1-r_n)(p_n.J)}{(1^T t_{\cdot n})^2}\left[AT_{\cdot n}\right]$$

where $T_{\cdot n}$ is the negative of $t_{\cdot n}$ except for the m^{th} row where $T_{m,n} = 1^T t_{\cdot n} - t_{m,n}$

Block 4:

The $\partial G^3/\partial j_m$ has the form (3) which gives

$$\frac{\partial G^3}{\partial j_m} = AFK_{\cdot m} + \sum_{i=1}^{k} \frac{(1-r_i)}{1^T t_{\cdot i}}\left[At_{\cdot i}(p_i.K_{\cdot m})\right]$$

where $K_{\cdot m}$ is a column vector with a 1 in the m^{th} row. The column has the same dimension as J.

Block 5:

For $\partial G^3/\partial r_m$ we have

$$\frac{\partial G^3}{\partial r_m} = -\frac{(p_m.J)}{1^T t_{\cdot m}} At_{\cdot m}$$

These rules hold for all cases except for blocks 2, 3, 4, and 5 when the numerator and denominator of the fraction goes to zero. For this case we need L'Hospital's Rule which can be stated as follows [10, page 110].

L'Hospital's Rule:

Let f(x) and g(x) represent functions which are differentiable at each point of a neighborhood $N_\epsilon(a)$. Assume $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \epsilon N'_\epsilon(a)$.

If

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

also exists and

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

We will need the following relationship

$$1^T t_{\cdot i} = p_{i \cdot} 1$$

Block 2:

$$\frac{\partial G^3}{\partial p_{ij}} = \frac{(1-r_i)}{1^T t_{\cdot i}} \left[A t_{\cdot i} j_j \right]$$

Now consider

$$\frac{\partial}{\partial t_{ki}} \left[\frac{(1-r_i)}{1^T t_{\cdot i}} [A t_{\cdot i} j_j] \right]$$

Let $f(t_{ki}) = (1-r_i) j_j a_{\cdot k} t_{ki}$

and $g(t_{ki}) = t_{ki}$.

Note that when $t_{ki} \to 0$, $f(0) = g(0) = 0$.

Now apply L'Hospital's Rule to give

$$f'(t_{ki}) = (1-r_i) j_j a_{\cdot k}$$

$$g'(t_{ki}) = 1$$

and

$$\lim_{t_{ki} \to 0} \frac{f'(t_{ki})}{g'(t_{ki})} = (1-r_i) j_j \; a_{\cdot k}$$

Block 3:

$$\frac{\partial G^3}{\partial t_{ij}} = \frac{(1-r_j)(p_{j\cdot}J)A(-t_{\cdot j})}{(1^T t_{\cdot j})^2} + \frac{(1-r_j)(p_{j\cdot}J)a_{\cdot i}(1^T t_{\cdot j})}{(1^T t_{\cdot j})^2}$$

Now consider

$$\text{(A)} \quad \frac{\partial}{\partial t_{\bar{K}j}} \left[\frac{(1-r_j)(p_{j\cdot}J)A(-t_{\cdot j})}{(1^T t_{\cdot j})(1^T t_{\cdot j})} \right]$$

and

$$\text{(B)} \quad \frac{\partial}{\partial p_{jK}} \left[\frac{(1-r_j)(p_{j\cdot}J)a_{\cdot i}}{p_{j\cdot}1} \right]$$

Taking (B) first, let

$$f(p_{jK}) = (1-r_j)p_{jK}j_K a_{\cdot i}$$

and $g(p_{jK}) = p_{jK}$.

Then

$$f'(p_{jK}) = (1-r_j)j_K a_{\cdot i}$$

$$g'(p_{jK}) = 1$$

and

$$\lim_{p_{jK}\to 0} \frac{f'(p_{jK})}{g'(p_{jK})} = (1-r_j)j_K a_{\cdot i}$$

Now re-write (A) as

$$\frac{\partial}{\partial t_{\bar{K}j}} \left[\frac{(1-r_j)(p_{j\cdot}J)}{p_{j\cdot}1} \cdot \frac{A(-t_{\cdot j})}{1^T t_{\cdot j}} \right]$$

Let $p_{jK} = t_{\bar{K}j} = \epsilon$ since $\lim_{\epsilon\to 0}[f(\epsilon)\cdot g(\epsilon)] = \lim_{\epsilon\to 0} f(\epsilon)\cdot\lim_{\epsilon\to 0} g(\epsilon)$

We have

$$\lim_{p_{jK}\to 0} = (1-r_j)j_K$$

and

$$\lim_{t_{\bar{K}j}\to 0} = -a_{\cdot\bar{K}}$$

Combining (A) and (B) gives

$$(1-r_j)j_K(a_{\cdot i} - a_{\cdot\bar{K}})$$

Block 4:

$$\frac{\partial G^3}{\partial j_m} = AFK_{\cdot m} + \sum_{i=1}^{k}\frac{(1-r_i)}{1^T t_{\cdot i}}\left[At_{\cdot i}(p_{i\cdot}K_{\cdot m})\right]$$

Now consider

$$\frac{\partial}{\partial t_{ji}}\left[\sum_{i=1}^{k}\frac{(1-r_i)}{1^T t_{\cdot i}}\left[At_{\cdot i}(p_{i\cdot}K_{\cdot m})\right]\right]$$

Here we have to consider each term separately.

Let $f(t_{ji}) = (1-r_i)p_{im}a_{\cdot j}t_{ji}$

and $g(t_{ji}) = t_{ji}$.

Apply L'Hospital's Rule to give

$$f'(t_{ji}) = (1-r_i)p_{im}a_{\cdot j} = 0$$

$$g'(t_{ji}) = 1$$

and $\lim_{t_{ji}\to 0} \frac{f'(t_{ji})}{g'(t_{ji})} = 0$

Block 5:

$$\frac{\partial G^3}{\partial r_i} = \frac{-(p_i.J)At_{.i}}{1^T t_{.i}}$$

Now consider

$$\frac{\partial}{\partial t_{ji}}\left[\frac{-(p_i.J)At_{.i}}{1^T t_{.i}}\right]$$

Let $f(t_{ji}) = -(p_i.J)a_{.j}t_{ji}$

and $g(t_{ji}) = t_{ji}$

Apply L'Hospital's Rule to give:

$$f'(t_{ji}) = -(p_i.J)a_{.j}$$

$$g'(t_{ji}) = 1$$

and $\lim\limits_{t_{ji}\to 0} \frac{f'(t_{ji})}{g'(t_{ji})} = -(p_i.J)\, a_{.j} = 0$

As an example of how these rules work, suppose that we construct column 4 of the underlying array of the pilot problem. To begin we need to know in which block variable 4 is located. Since we have five polluters and three sections, block 1 has $5 \cdot 3 = 15$ columns and we are in block 1. The formulas for recovering the row and column are

$$i = \frac{N_v - 1}{N_p} + 1 \quad \text{and} \quad j = N_v - \left[\frac{N_v - 1}{N_p} \times N_p\right]$$

(Note use integer division)

where

N_v is the variable number

N_p is the number of polluters

Thus,

$$i = \frac{4 - 1}{5} + 1 = 1 \quad \text{and} \quad j = 4 - [0 \times 5] = 4$$

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} & \boxed{f_{14}} & f_{15} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} \end{bmatrix}$$

Applying the rules we have

$$\partial G^1/\partial f_{14} = K^T_{4,1} \cdot 1$$

$$\frac{\partial G^1}{\partial f_{14}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \frac{\partial G^2}{\partial f_{14}} = \odot = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial G^3}{\partial f_{14}} = AK_{14}J \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}\, j_4 \\ a_{21}\, j_4 \\ a_{31}\, j_4 \end{bmatrix}$$

and the completed column is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_{11}\, j_4 \\ a_{21}\, j_4 \\ a_{31}\, j_4 \\ k_{14} \end{bmatrix}$$

where k_{14} is the cost coefficient. In this example note that in calculating the $\partial G^3/\partial f_{14}$ we multiplied the A matrix against $K_{14}J$. Essentially we are taking the inner product of each row of the A matrix against the column vector $K_{14}J$. In fact, every block of the quality constraints can be reduced to a multiplication of the A matrix against a column vector! When we consider the size of the underlying array of the Delaware Estuary problem, and the number of columns generated to solve one local L.P., we can get some idea of the tremendous number of inner products computed in solving a problem of this size. A scheme for computing inner products more efficiently has been given by Winograd [11].

Let $X = (x_1, x_2, x_3, x_4, x_5, x_6)$

and let $Y = [y_1, y_2, y_3, y_4, y_5, y_6]$.

The standard computation of $X \cdot Y$ is

$$X \cdot Y = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 + x_5 y_5 + x_6 y_6$$

which has six multiplications and 5 additions. Now suppose we calculate $X \cdot Y$ as follows

$$\begin{aligned} X \cdot Y = {} & (x_1 + y_2)(x_2 + y_1) + (x_3 + y_4)(x_4 + y_3) + \\ & (x_5 + y_6)(x_6 + y_5) - (x_1 x_2 + x_3 x_4 + x_5 x_6) - \\ & (y_1 y_2 + y_3 y_4 + y_5 y_6) \end{aligned}$$

At first glance it appears that we have lost ground. But if the last two terms have been pre-calculated and the results stored, we have only three multiplications and ten additions. In terms of machine computation, we have gained ground because we halved the multiplications while doubling the additions. The formulas used in this procedure are as follows.

Let $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n)$

$$\xi = \sum_{j=1}^{n/2} x_{2j-1}\, x_{2j} \qquad \eta = \sum_{j=1}^{n/2} y_{2j-1}\, y_{2j}$$

$$\gamma = \sum_{j=1}^{n/2} (x_{2j-1} + y_{2j})(x_{2j} + y_{2j-1})$$

The inner product $X \cdot Y$ is then given by

$$X \cdot Y = \begin{cases} \gamma - \xi - \eta & \text{if } n \text{ is even} \\ \gamma - \xi - \eta + x_n y_n & \text{if } n \text{ is odd} \end{cases}$$

We make use of this procedure by noting that in the expressions for the partial derivatives of the quality constraints we can factor the A matrix out of each expression. Thus we pre-calculate all the row products for the A matrix and store the sum. The remainder of the expression is reduced to a column vector and the column products are calculated and the sum is saved. Then we apply the above formulas to get the inner product.

We can also make use of this inner product routine in evaluating the quality constraints, G^3. Recall from Section II that in setting up the local L.P., we evaluate $G^3(y^o)$ for the right hand sides. The quality constraints are

$$G^3 = AFJ + A \sum_{i=1}^{k} \frac{t_{\cdot i}(1-r_i)(p_{i\cdot}J)}{1^T t_{\cdot i}} - A\bar{F}\bar{J} + C \leq 0$$

Factoring out A gives

$$G^3 = A\left[FJ + \sum_{i=1}^{k} \frac{t_{\cdot i}(1-r_i)(p_{i\cdot}J)}{1^T t_{\cdot i}} - \bar{F}\bar{J} \right] + C \leq 0.$$

Note that the expression in brackets reduces to a column vector.

V. DEVELOPMENT OF TREATMENT COSTS

Treatment Costs at the Polluter:

The costs to polluters to reduce BOD discharges were determined by the Delaware Estuary Comprehensive Study Group through direct questioning of the officials of the industrial firms and municipal treatment plants. They were asked to determine how much it would cost to reduce BOD discharges by the year 1975 [12, page 50].

The data collected included present discharges and the cost to reduce BOD discharges (nearly) to zero. Thus, each polluter provided either one, two, or three points on his waste reduction cost curve. That is, each polluter gave the cost of reducing waste discharges by one, two, or three different amounts.

The cost estimates provided by the polluters are the best cost estimates available, yet it is important to recognize their limitations. Engineering estimates of costs cannot be made with complete confidence, and as a result there may be significant errors in the cost data. It may also be the case that some of the polluters biased their estimates in an effort to protect their interests. Since process changes were considered as an alternative to waste treatment in the estimation of costs, the inherent difficulty of estimating the cost of process changes is involved.

The data we used in this study is given in Table 9 .

TABLE 9

Polluter Cost and Discharge Data

Polluter Number	Slope s_i \$/lb/day	Increment Bound B_i lb/day	Flow f_i MGD	Present Discharge $\bar{U}_i$ lb/day
1	5980	2040	19.7	3060
2	3769	1133	6.1	1700
3	2455	1833	250.	2750
4	1843 10702	346 228	15.2	990
5	1749	2093	4.6	3140
6	105 4809	1333 445	3.0	2000
7	59 1051	5808 1161	2.0	7550
8	191 2735	892 892	4.0	2230
9	33 192	50100 35070	144.7	125250
10	66 91	42507 8501	22.4	59510
11	149 1452	9712 1942	7.0	12605
12	199 285	130854 18319	107.6	170110
13	14 498	12238 2448	3.5	15910
14	1836 1471	1780 214	10.6	3560
15	250 4157	880 587	4.9	1760

TABLE 9 CON'T.

Polluter Number	Slope s_i \$/lb/day	Increment Bound B_i lb/day	Flow f_i MGD	Present Discharge $\bar{U}_i$ lb/day
16	247	20672	28.2	21925
17	258	9923	51.0	12900
	3416	1985		
18	559	3346	6.8	3955
	6230	305		
19	135	1552	2.0	2070
	4377	345		
20	232	117150	118.0	156200
	291	18223		
21	123	1335	0.7	1870
	1828	267		
22	346	9471	38.6	25650
	724	8287		
	6488	1973		
23	177	5091	0.3	8100
	324	1389		
	324	1389		
24	198	27160	80.4	34160
	1108	4000		
	3235	2000		
25	3474	2107	11.6	3160
26	3403	717	4.0	1075
27	126	1350	0.7	1890
	3630	270		
28	348	5365	6.6	7750
29	987	3745	14.8	7745
	2317	2000		
	3791	1400		
30	252	7275	8.6	10185
	4192	1455		

TABLE 9 CON'T.

Polluter Number	Slope s_i \$/lb/day	Increment Bound B_i lb/day	Flow f_i MGD	Present Discharge $\bar{U}_i$ lb/day
31	263 872 2912	1219 1273 831	112.2	3600
32	487	2855	5.4	3145
33	107 3286	1400 280	1.4	1820
34	198 410 796	19590 2857 3265	1.1	32650
35	384 858 2900	8044 16089 3448	106.9	28730
36	803 7197	1445 289	33.3	2890
37	171 359	53732 8059	60.0	85970
38	140 3067	1073 238	1.0	1430
39	3040 5055	2500 2750	4.8	8480
40	469 686	74000 30500	103.3	110000
41	381	5066	10.1	6755
42	107 2340	1402 312	1.2	1870
43	458 15844	1923 385	278.6	2500
44	116 2535	1298 288	1.3	1730

As an example of the meaning of this data, consider the polluter number 23 which supplied three points on the cost curve. This polluter is currently discharging 8100 lbs/day of BOD. For the meaning of the slope and upper bound, see Figure 7. The costs in Figure 7 are the present value of the construction and operating costs. They were computed using a present value factor of 13 by the following formula:

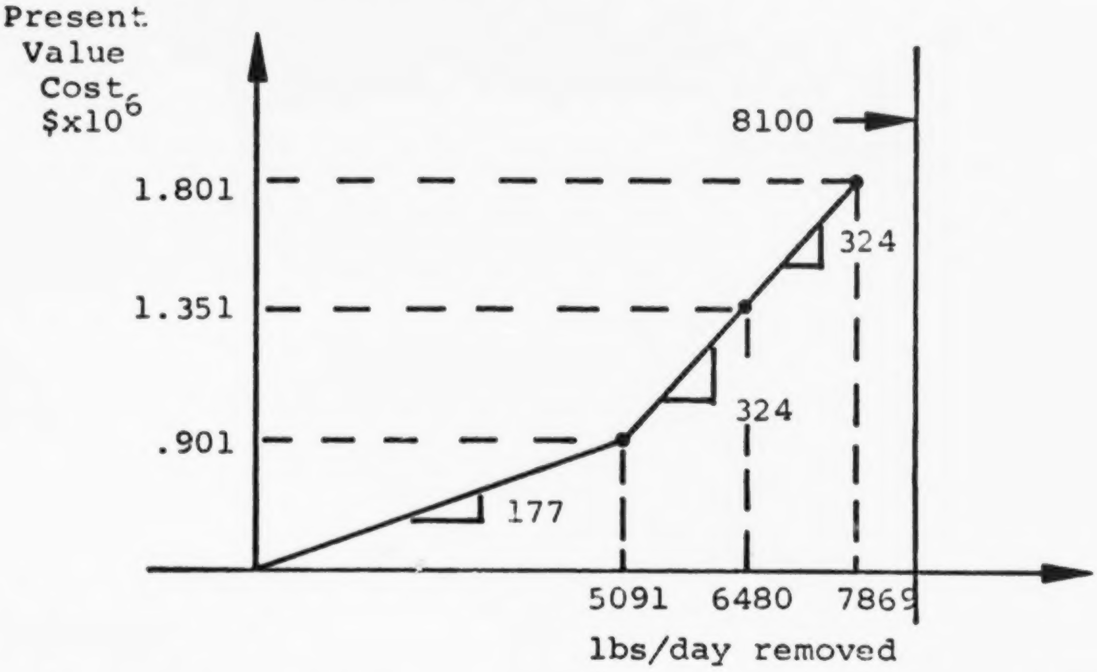
$$(\text{Total cost})_{\text{P.V.}} = \text{cap. cost} + 13(\text{op. cost})^{1}$$

To make these costs compatible with our piping costs which are annual costs, we make the following conversion:

$$(\text{Total cost})_{\text{ANN.}} = \frac{\text{cap. cost}}{13} + \text{op. cost}$$

FIGURE 7

Piecewise Linear Cost Curve



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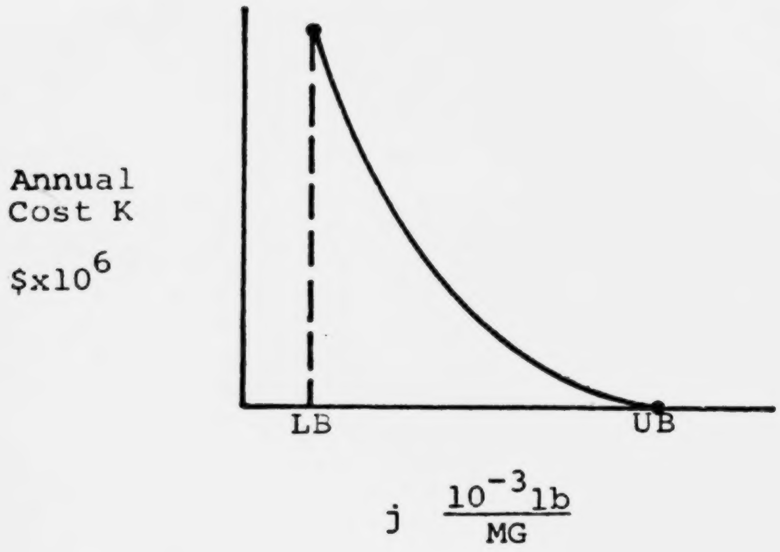
[1] See [12] pg. 52. The present value factor of 13 is based on a discounting period of 20 years with a discount rate between 3% and 5% per annum.

For our purposes, the above data is not in a desirable form . We need an expression for the cost as a function of the variable j. Initially, we used the general expression,

$$K = c_1(UB-j)^{c_2} \quad \text{where } K = \text{cost},$$

to fit the data points supplied by that polluter (See Figure 8).

FIGURE 8
Smooth Curve Fit of Cost Data



However, this expression was found to be undesirable because the fitting introduced local minimums near the upper bound. Thus, most of the polluters moved slightly off their upper bounds and a large amount of calculating time was wasted making slight improvements which were meaningless.

The most satisfactory cost function was found to be of the simple piecewise linear type. No external constraints were added and all the algebra and logic were done internally in the machinery of the algorithm.

Using a piecewise linear cost function, we define for each polluter with three price breaks:

$$J1 = \frac{\bar{U}}{\bar{f}} \times 10^{-3}, \qquad J2 = \frac{\bar{U} - B1}{\bar{f}} \times 10^{-3}$$

$$J3 = \frac{\bar{U} - B1 - B2}{\bar{f}} \times 10^{-3}, \quad J4 = \frac{\bar{U} - B1 - B2 - B3}{\bar{f}} \times 10^{-3}$$

$$k_1 = s^1 \bar{f} \times 10^{-3}/13, \quad k_2 = s^2 \bar{f} \times 10^{-3}/13, \quad k_3 = s^3 \bar{f} \times 10^{-3}/13$$

If we have two breaks we simply omit J4 and k_3 and if we have one price break we omit J4, J3, k_3 and k_2. The cost function becomes

$$K = (J1 - j) \cdot k_1 \qquad J2 < j \leq J1$$

$$K = (J1 - J2) \cdot k_1 + (J2 - j) \cdot k_2 \qquad J3 < j \leq J2$$

$$K = (J1-J2) \cdot k_1 + (J2-J3) \cdot k_2 + (J3-j) \cdot k_3 \qquad J4 \leq j \leq J3$$

The cases with one, two and three price breaks are plotted in Figures 9, 10, 11.

FIGURE 9

One Price Break

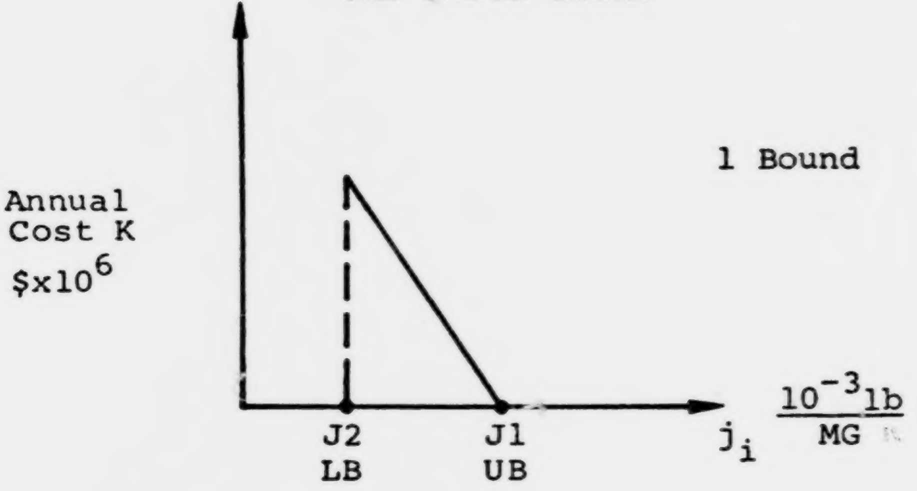

1 Bound
Annual Cost K $\$ \times 10^6$
J2 LB
J1 UB
j_i
$\frac{10^{-3}\,lb}{MG}$

FIGURE 10

Two Price Breaks

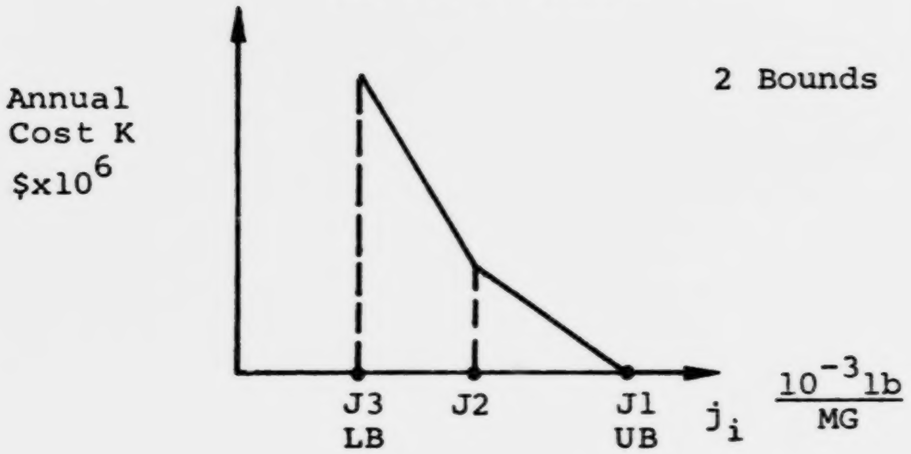
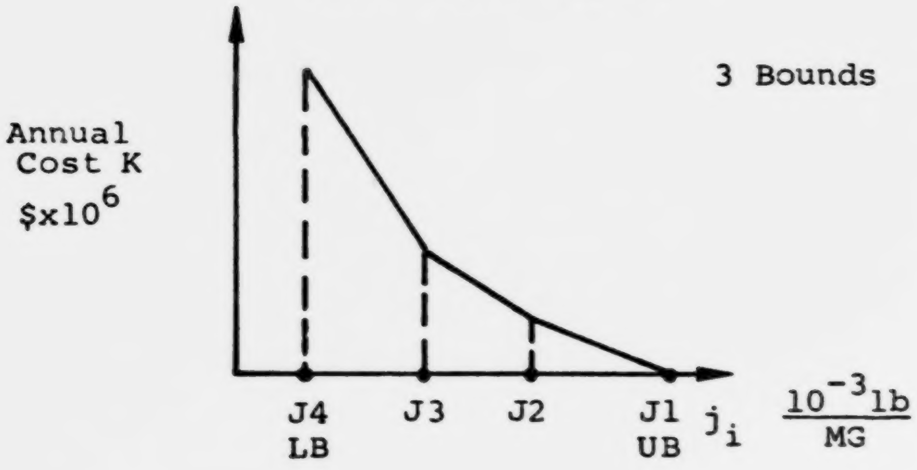

2 Bounds
Annual Cost K $\$ \times 10^6$
J3 LB
J2
J1 UB
j_i
$\frac{10^{-3}\,lb}{MG}$

FIGURE 11

Three Price Breaks


3 Bounds
Annual Cost K $\$ \times 10^6$
J4 LB
J3
J2
J1 UB
j_i
$\frac{10^{-3}\,lb}{MG}$

The partial derivatives of the cost function for each of the three cases are as follows:

Case 1: $J2 \leq j_i \leq J1$ $\quad \frac{\partial K}{\partial j_i} = -s_i^1 \bar{f}_i \cdot (10^{-3}/13)$

Case 2: $J3 \leq j_i \leq J2$ $\quad \frac{\partial K}{\partial j_i} = -s_i^2 \bar{f}_i \cdot (10^{-3}/13)$

Case 3: $J4 \leq j_i \leq J3$ $\quad \frac{\partial K}{\partial j_i} = -s_i^3 \bar{f}_i \cdot (10^{-3}/13)$

Cost of treatment at a regional plant:[1]

The variables corresponding to regional treatment plants are the r_i (the fractional removal of BOD at the ith regional plant). The costs of BOD removal as a function of the percent removed have been given by [13, page 10]. Costs were given in terms of total annual costs of sewage treatment per MGD capacity based on 20 year, 4½% municipal bonds covering the total initial capital investment plus the annual operation and maintenance costs of the treatment plant. Costs are representative of plants designed and operated in humid areas where water scarcity is not a problem. These costs are given in Figure 12 with the design capacity as a parameter for four different sized plants. The costs were fitted to an equation of the form

$$K_c = a(w - .5)^3 + b$$

The equation was parameterized to yield the following general expression

$$K_c = \frac{49.22}{Q^{1/4}}\left[8.0(w - .5)^3 + 1\right] \qquad (40)$$

where

K_c is the total annual cost per MGD in \$1000

w is the fractional removal of BOD

Q is the design capacity in MGD.

As an example of the economies of scale obtained by using regional treatment, consider the following situation. Suppose four polluters, each having a flow of exactly 2.5MGD decided to build treatment plants to obtain 50% removal. From Figure 12 we see that the total annual cost would be

$$38{,}000\left(\frac{\$}{\text{MGD}}\right) \times 2.5 \text{ MGD} \times 4 = \$380{,}000$$

If the four polluters were located close enough so that piping costs would not be excessive, they might consider building only one plant and sharing the cost. From Figure **12**, the total annual cost would be

$$26{,}000\left(\frac{\$}{\text{MGD}}\right) \times 10 \text{ MGD} = \$260{,}000$$

which is a \$120,000 savings.

Theoretically, the cost curves go to infinity as $w \rightarrow 1.0$. To cope with this physical reality and its mathematical consequences, we restrict $w \leq .98$. The bound can be chosen to reflect the accuracy of the cost data. Before presenting the scheme for computing costs at the regional plants, it

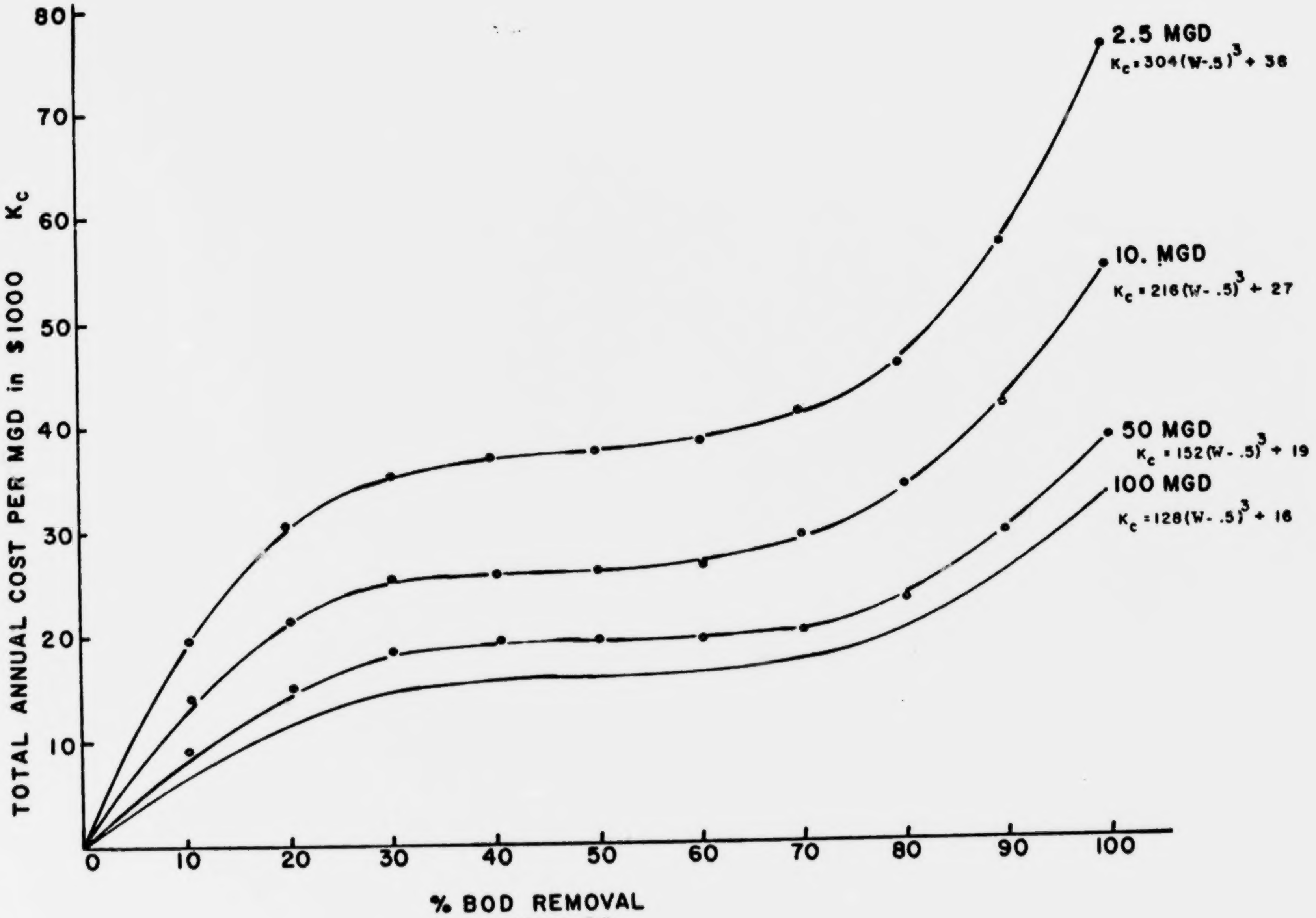


FIGURE 12
Plant Costs

might be interesting to compare annual costs at the polluters with those at the treatment plants. Notice that equation (40) is an efficient tool for making this comparison.

At polluter 1 we have:

f_1 = 19.7 MGD B_1 = 2040 lbs/day
$\bar{j}_1$ = 18.6 mg/l s_1 = 5980 \$/lb/day
$\bar{U}_1$ = 3060 lbs/day
$\bar{q}_1$ = .85 (fractional removal at polluter 1)

If we removed all 2040 lb/day, the cost would be

$$\frac{5980}{13} \times 2040 = \$938,400$$

with $U_1 = 3060 - 2040 = 1020$ lb/day

$$j_1 = \frac{1020}{19.7 \cdot 8.33} = 6.2 \text{ mg/l}$$

and

$$q_1 = 1 - \frac{6.2}{124} = .95$$

Note that $\bar{\bar{j}}_1 = \frac{18.6}{(1-.85)} = 124$ mg/l (untreated concentration)

Now evaluate (40) with Q = 19.7 and w = .85 and .95.

w = .85

$$K_c = \frac{49.22}{19.7^{1/4}}\left[8.0(.85-.5)^3 + 1\right] = 31.376$$

and the cost is 31,376·19.7 = \$618,111.

w = .95

$$K_c = \frac{49.22}{19.7^{1/4}}\left[8.0(.95-.5)^3 + 1\right] = 40.394$$

and the cost is 40,394·19.7 = \$795,767.

The first cost \$618,111 is the present annual cost to maintain treatment at the 85% level. The second cost \$795,767 is the annual cost of constructing and operating a new plant that would give 95% removal. The estimate given by polluter 1 is \$938,400. We conlude from the magnitude of this figure and its closeness to the theoretical value fo \$795,767 that this cost is not for additional units. It must be the cost of a complete new plant to give 95% removal.

This same type of comparison was made for all 44 polluters and the results are given in Table 10. Column two gives the classification of the type of waste, i.e., domestic or industrial. Column three gives the untreated concentration in (mg/l). Columns four, five and six give the linear costs versus the costs computed from (40) for different levels of per cent removal at the polluter. Note that the comparison is made for the 22 industrial wastes although equation (40) is not valid for plants treating industrial wastes. The conclusion from Table 10 is that the costs are mixed, i.e., some costs are for new plants. Thus the cost data are inconsistent in the sense that two identical polluters can have widely different costs to achieve the same increase in treatment.

Note here that we are using a translated set of w's in the sense that the fractional removal depends on the point of view. In the above example, we could say that since the treatment level was increased from 85% to 95% the increase in treatment was 10%. However, we didn't know this fact we say that since 2040 of the 3060 lbs/day were removed.

TABLE 10

Comparisons of Estimated Plant Cost

POLLUTER NUMBER	TYPE WASTE	mg/l	PER CENT BOD REMOVAL	COST EQN. 1	COST EST. OF POLLUTER
1	D	124.	0.85	618111.	
			0.95	795767.	938400.
2	D	223.	0.85	256575.	
			0.95	330292.	328483.
3	I	9.	0.85	4155968.	
			0.95	5350188.	346155.
4	I	8.	0.0	0.	
			0.35	368565.	49052.
			0.58	380439.	236749.
5	I	546.	0.85	207628.	
			0.95	267291.	281589.
6	D	178.	0.55	112309.	
			0.85	150656.	10767.
			0.95	194016.	175382.
7	I	697.	0.35	80543.	
			0.85	111177.	26359.
			0.95	143115.	120222.
8	D	268.	0.75	156617	
			0.85	186966.	13106.
			0.95	240703.	200769.
9	D	416.	0.75	2310178.	
			0.85	2757838.	127177.
			0.92	3270604.	645134.
10	D	456.	0.30	474354.	
			0.80	616253.	215805.
			0.90	766251.	275312.
11	D	333.	0.35	206100.	
			0.85	284487.	111314.
			0.95	366243.	328221.
12	D	292.	0.35	1599974.	
			0.85	2208395.	2003072.
			0.92	2618988.	2404680.
13	I	840.	0.35	122548.	
			0.85	169142.	13179.
			0.95	217761.	106957.
14	I	40.	0.0	0.	
			0.50	289148.	251391.
			0.56	289651.	275606.
15	D	144.	0.70	172476.	
			0.85	217703.	16923.
			0.95	280319.	204628.
16	I	267.	0.65	618584.	
			0.98	1135208.	392768.
17	I	47.	0.35	913970.	
			0.85	1261511.	196933.
			0.95	1624174.	718530.

TABLE 10 CON'T.

18	I	107.	0.35	201667.	
			0.90	313312.	143878.
			0.95	358397.	290043.
19	D	207.	0.40	82116.	
			0.85	111135.	16117.
			0.95	143065.	132276.
20	D	265.	0.40	1748091.	
			0.85	2366618.	2090676.
			0.92	2806635.	2498590.
21	D	458.	0.30	35257.	
			0.80	45782.	12631.
			0.90	56907.	50175.
22	I	123.	0.35	741643.	
			0.59	766668.	252074.
			0.80	926876.	713596.
			0.85	1023678.	1698275.
23	I	9261.	0.65	20491.	
			0.87	28036.	69316.
			0.93	32642.	103934.
			0.99	38732.	138552.
24	I	78.	0.35	1285866.	
			0.87	1843309.	413668.
			0.94	2240168.	754591.
			0.98	2497884.	1252283.
25	D	218.	0.85	415490.	
			0.95	534932.	563055.
26	D	215.	0.85	186966.	
			0.95	240734.	187689.
27	I	463.	0.30	35257.	
			0.80	45803	13085.
			0.90	56953.	88477.
28	D	217.	0.35	197202.	
			0.80	246438.	143617.
29	I	157.	0.60	374367.	
			0.79	446450.	284332.
			0.90	556894.	640793.
			0.97	677931.	1049054.
30	D	203.	0.30	231361.	
			0.80	300572.	141023.
			0.90	373738.	610205.
31	I	6.	0.35	1651005.	
			0.57	1701495.	24661.
			0.80	2063128.	110050.
			0.95	2933687.	296194.
32	I	108.	0.35	169648.	
			0.94	293226.	106953.
33	D	240.	0.35	61638.	
			0.85	85077.	11523.
			0.95	109530.	82298.
34	I	4454.	0.20	41448.	
			0.68	55334.	298371.
			0.75	59476.	388476.
			0.83	68067.	588395.

TABLE 10 CON'T.

35	I	129.	0.75	1840888.	
			0.82	2065289.	237607.
			0.96	2910532.	1299481.
			0.99	3176476.	2068650.
36	I	10.	0.0	0.	
			0.50	682299.	89257.
			0.60	687757.	249251.
37	D	215.	0.20	831899.	
			0.70	1129011.	706782.
			0.78	1237634.	929335.
38	D	286.	0.40	48826.	
			0.85	66133.	11555.
			0.95	85118.	67705.
39	D	212.	0.0	0.	
			0.29	148583.	584615.
			0.62	161772.	1653941.
40	I	197.	0.35	1551777.	
			0.79	1897312.	2669692.
			0.97	2898458.	4279152.
41	I	401.	0.80	339091.	
			0.95	482134.	148473.
42	D	312.	0.40	55981.	
			0.85	75762.	11540.
			0.95	97557.	67700.
43	I	2.	0.35	3265802.	
			0.85	4507479.	67749.
			0.95	5804563.	536975.
44	D	266.	0.40	59444.	
			0.85	80508.	11582.
			0.95	103625.	67742.

the treatment increase was 66%. The relation between the translated variables and the r variables is

$$r_i = \frac{w_i - \bar{w}_i}{1 - \bar{w}_i} \tag{41}$$

To compute costs at a regional plant, we will make two evaluations of equation (40) and take the difference as the cost. At the ith regional plant, we note that $Q = \sum_j p_{ij}$ in (40) and we calculate a $\bar{v}_i$ and a v_i as follows

$$\bar{v}_i = \sum_j \frac{p_{ij}(\bar{\bar{j}}_j - j_j)}{\sum_j p_{ij}\,\bar{\bar{j}}_j}$$

and $\bar{v}_i \leq v_i \leq .98$. Thus, we have

$$\bar{v}_i = 1 - \sum_j \frac{p_{ij} j_j}{\sum_j p_{ij}\bar{\bar{j}}_j} \tag{42}$$

If $\sum_j p_{ij} = 0$, (42) is indeterminate. In this case, the value of $\bar{v}_i$ is irrelevant since the cost equation will automatically go to zero. From (41) we have

$$v_i = r_i + \bar{v}_i - r_i\bar{v}_i \tag{43}$$

The cost equation is now written as

$$\text{Cost}(\$\text{x}10^6) = \sum_{i=1}^{k} .04922\,(\sum_j p_{ij})^{3/4}[(8.0(v_i - .5)^3 + 1)$$

$$- (8.0(\bar{v}_i - .5)^3 + 1)]$$

which can be simplified to

$$\text{Cost}(\$\text{x}10^6) = \sum_{i=1}^{k} .39376 (\sum_j p_{ij})^{3/4} [(v_i - .5)^3 - (\bar{v}_i - .5)^3] \qquad (44)$$

with v_i and $\bar{v}_i$ given by (42) and (43).

Computing costs in this manner assumes that at a regional plant we build only the additional units that we need to achieve the desired per cent removal. With the economy of scale effect from combining flows and using only the cost of additional units, the costs at the regional plants should be competitive with those at the polluters.

The respective partial derivatives of the cost function are as follows:

$$\frac{\partial}{\partial r_i} = 1.18128 \; (\sum_j p_{ij})^{3/4} \; (v_i - .5)^2 \; (1 - \bar{v}_i)$$

$$\frac{\partial}{\partial j_k} = 1.18128 \sum_{i=1}^{k} \left[(\sum_j p_{ij})^{3/4} \cdot \frac{p_{ik}}{\sum_j p_{ij} \bar{\bar{j}}_j} \left([\bar{v}_i - .5]^2 - (1 - r_i) [v_i - .5]^2 \right) \right]$$

$$\frac{\partial}{\partial p_{ik}} = 1.18128 \left(\frac{(\sum_j p_{ij})^{-1/4}}{4} [(v_i - .5)^3 - (\bar{v}_i - .5)^3] + (\sum_j p_{ij})^{3/4} \left[\frac{\sum_j p_{ij} (j_j \bar{\bar{j}}_k - \bar{\bar{j}}_j j_k)}{(\sum_j p_{ij} \bar{\bar{j}}_j)^2} \right] [(1 - r_i) \; (v_i - .5)^2 - (\bar{v}_i - .5)^2] \right)$$

If site costs are known for each of the regional plants, we can include the relative differences of these costs in (44) as follows:

$$\text{Cost}(\$\text{x}10^6) = \sum_{i=1}^{k} .39376(\sum_j p_{ij})^{3/4}[(v_i-.5)^3 - (\bar{v}_i-.5)^3]\cdot S$$

where S is greater or less than one depending upon the ratio of site cost to some standard site cost. This effect can differentiate between two plants located extremely close together.

By-pass piping costs:

The piping costs associated with each flow variable are given by the general expression [7, page 23]

$$D_{ij}^{k}\ (\frac{\$}{\text{MGD - yr}}) = 1865\ d_{ij}^{k}\ Q_{ij}^{-.402} \qquad k=1,2,3$$

where

Q_{ij} is the flow in MGD and corresponds to a particular f_{ij}, p_{ij}, or t_{ij}.

The cost function for this problem is made up of piping costs and treatment costs at the polluters and at the regional plants. The entire cost function can now be assembled and is given as follows.

$$\sum_{ij} D_{ij}^{1} f_{ij} + \sum_{ij} D_{ij}^{2} p_{ij} + \sum_{ij} D_{ij}^{3} t_{ij} + .39376 \sum_i \left[(\sum_j p_{ij})^{3/4} [(v_i-.5)^3-(\bar{v}_i-.5)^3]\right] + \text{piecewise linear costs at polluters}$$

VI. A SMALL SCALE PROBLEM

Recall that in Section III, we set up a small scale problem with 5 polluters, 3 sections, and 3 potential regional treatment plants (See Figure 6 and Table 12). Recall that the j variables are bounded above and below (See Section V). The r variables are absolutely bounded, of course, by 0.0 and 1. However, in order to control the magnitudes of the v_i (See Section V) and maintain stability of the cost function, we will restrict r so that $0 \leq r_i \leq 0.70$. Note that from (41) in the previous section, the real removal rate would be

$$w_i = r_i \cdot (1 - \bar{w}_i) + \bar{w}_i .$$

When we have 95% removal at a polluter followed by 70% removal at the regional plant, the total removal would be

$$w_i = .70 \cdot (1 - .95) + .95 = .985.$$

The first configuration that we will investigate has the variables arranged in the following priority classes:

TABLE 11

Priority Class
Configuration 1

1	$\{f_{11}, f_{12}, f_{23}, f_{24}, f_{35}, p_{11}, p_{13}, p_{15}, p_{22}, p_{24}, p_{31}, p_{33}, p_{35}, t_{12}, t_{21}, t_{33}, r_1, r_2, r_3\}$
2	$\{j_1, j_2, j_3, j_4, j_5\}$
3	$\{t_{11}, t_{13}, t_{22}, t_{23}, t_{31}, t_{32}\}$
4	$\{f_{13}, f_{14}, f_{15}, f_{21}, f_{22}, f_{25}, f_{31}, f_{32}, f_{33}, f_{34}\}$
5	$\{p_{12}, p_{14}, p_{21}, p_{23}, p_{25}, p_{32}, p_{34}\}$

TABLE 12

Polluter Data for Example

Polluter Number	Number (from Table 9)	f_{ij} (MGD)	$\bar{J}_i$ lb/MG	% removal BOD
1	1	19.7	155	85
2	11	7.0	1801	35
3	6	3.0	666	55
4	2	6.1	278	85
5	8	4.0	557	75

Note that in priority class 1, treatment is restricted to the regional plants. Activating class 2 forces us to choose between treatment at the polluter and treatment at the regional plants. Class 3 allows the regional plants to choose discharge sections and class 4 opens up all the by-pass pipes.

From Table 6 (See Section III), we see that there is a substantial negative goal in section 3. Thus, we might guess that when the by-pass pipes are opened up, there will be a dramatic shift in the solution. However, it should be recalled our cost function for the regional plants and by-pass pipes is non-convex and we are using a local method. This confines us to local optima and makes our solutions dependent on the order in which we introduce the variables. This is illustrated by the slight discrepancy in the solutions to Configuration 1 and 2 Priority Class 3 which we would expect to be identical.

The solution to priority class 1 is given in Figure 13 and Table 13.

FIGURE 13
Solution of Example Configuration 1 and Priority Class 1

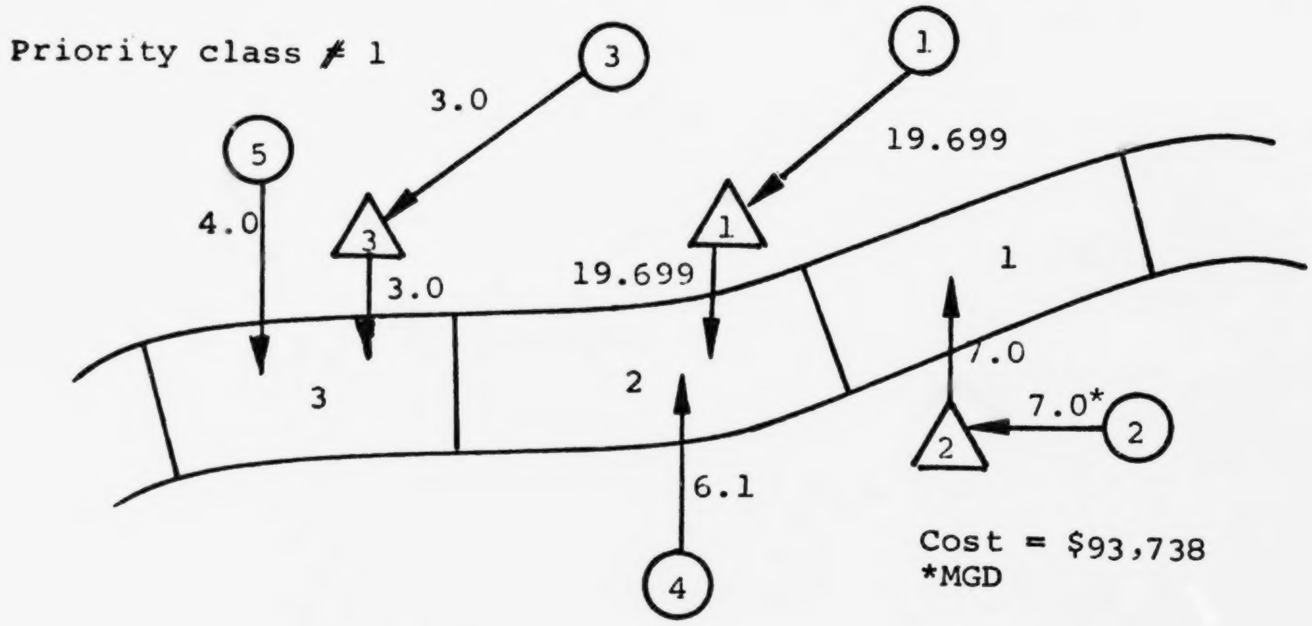

Priority class # 1
3
1
3.0
19.699
5
4.0
3
1
3.0
19.699
1
3
2
7.0
7.0*
2
2
6.1
4
Cost = $93,738
*MGD

TABLE 13
Solution Data for Example Configuration 1 and Priority Class 1

		% removal BOD
regional plant	1	0
	2	69.4
	3	8.69

Note from Table 13 that regional plant 1 is not opened and is nothing more than a junction point. Opening up priority class #2 gives no further gains.

The solution to priority class 3 is given in Figure 14 and Table 14.

FIGURE 14

Solution of Example Configuration 1 and Priority Class 3

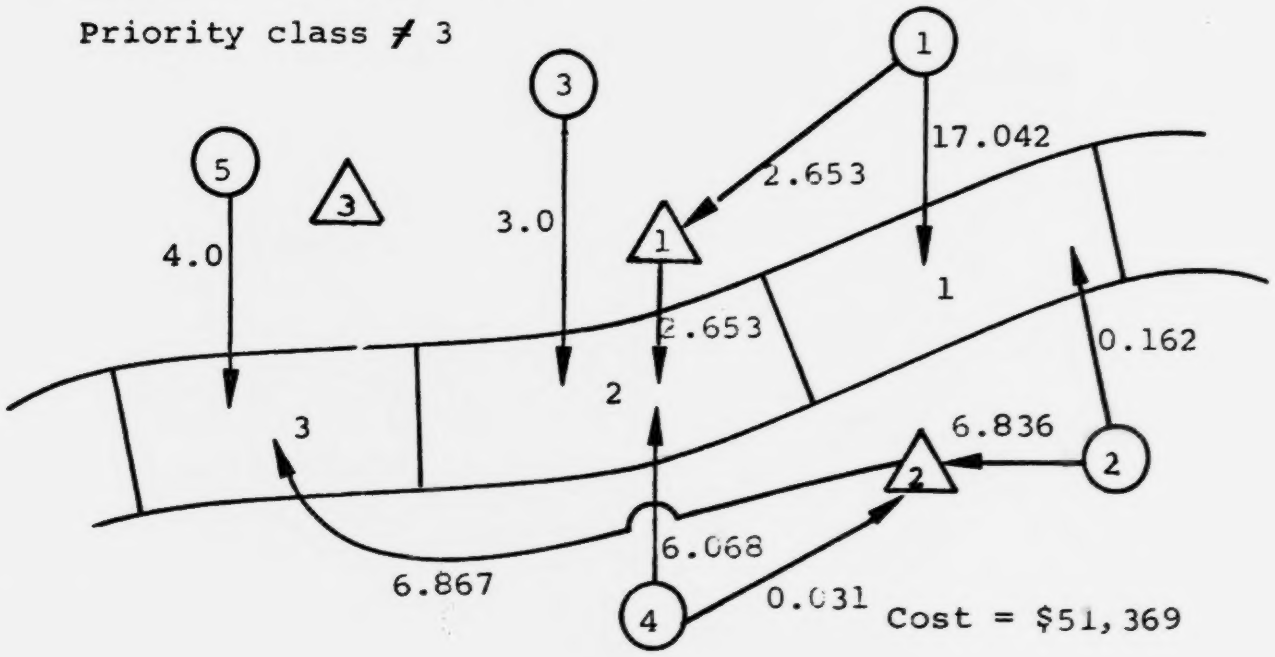

Priority class ≠ 3
1
3
5
3
17.042
2.653
4.0
3.0
1
2.653
1
0.162
2
3
6.836
2
2
6.068
6.867
0.031
4
Cost = $51,369

TABLE 14

Solution Data for Example Configuration 1 and Priority Class 3

	% removal BOD	Polluter Number	Lower Bound lb/MG	Concentration lb/MG	Upper Bound lb/MG	% removal BOD
regional plant 1	0	1	52	155	155	85
2	36.9	2	136	1801	1801	35
3	—	3	74	666	666	55
		4	93	278	278	85
		5	111	557	557	75

Notice that in this solution plant 1 does not treat and becomes a junction point for by-pass piping. This occurs because the negative goal in section 3 makes by-pass piping attractive and the by-pass flow variables remain in priority class 4. The best solution opening all priority classes was $44,997.

The second configuration we investigated had the variables arranged in the following priority classes:

TABLE 15

Priority Class
Configuration 2

1	$\{j_1, j_2, j_3, j_4, j_5\}$
2	$\{f_{11}, f_{12}, f_{23}, f_{24}, f_{35}\}$
3	$\{p_{11}, p_{13}, p_{15}, p_{22}, p_{24}, p_{31}, p_{33}, p_{35}, t_{11}, t_{12}, t_{13}, t_{21}, t_{22}, t_{23}, t_{31}, t_{32}, t_{33}, r_1, r_2, r_3\}$
4	$\{f_{13}, f_{14}, f_{15}, f_{21}, f_{22}, f_{25}, f_{31}, f_{32}, f_{33}, f_{34}\}$
5	$\{p_{12}, p_{14}, p_{21}, p_{23}, p_{25}, p_{32}, p_{34}\}$

The solution to priority class 1 is given in Table 16. Notice that this solution is pure treatment at the polluter with a cost of $180,843. Opening up class 2, of course, had no effect. The solution to class 3 is given in Figure 15 and Table 17.

TABLE 16

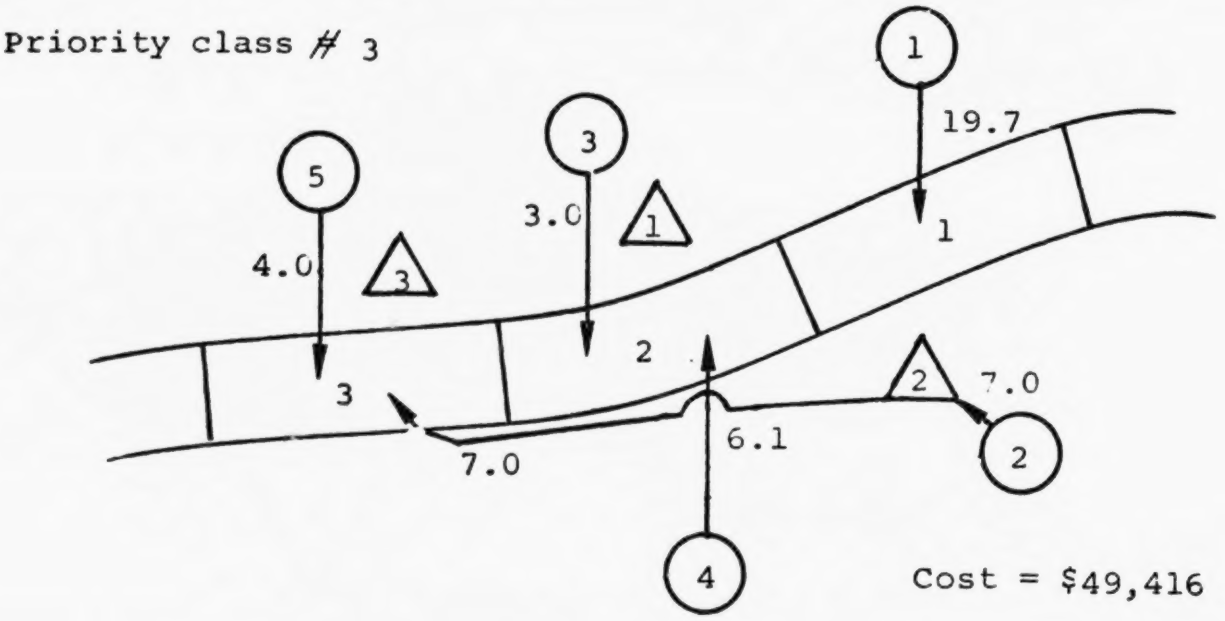
Solution Data for Example Configuration 2 and Priority Class 1

Polluter Number	Lower Bound lb/MG	Concentration lb/MG	Upper Bound lb/MG	% removal BOD
1	52	155	155	85
2	136	358	1801	87
3	74	222	666	85
4	93	278	278	85
5	111	334	557	85

Cost = $180,843

FIGURE 15

Solution of Example Configuration 2 and Priority Class 3



Cost = $49,416

TABLE 17

Solution Data for Example Configuration 2 and Priority Class 3

	% removal BOD	Polluter Number	Lower Bound lb/MG	Concentration lb/MG	Upper Bound lb/MG	%removal BOD
regional plant 1	—	1	52	155	155	85
2	38.9	2	136	1801	1801	35
3	—	3	74	666	666	55
		4	93	278	278	85
		5	111	557	557	75

Notice by comparing Tables 14 and 17 that in this solution we have a slightly higher treatment level at plant 2 than in the solution to Configuration 1 Priority Class 3. This enables polluters 1 and 4 to discharge their total flow directly into the nearest section and gives a slightly improved cost. The best solution opening all priority classes was $45,406.

The final configuration we investigated was the following:

TABLE 18

Priority Class Configuration 3

1	$\{f_{11}, f_{12}, f_{13}, f_{14}, f_{15}, f_{21}, f_{22}, f_{23}, f_{24}, f_{25}, f_{31}, f_{32}, f_{33}, f_{34}, f_{35}\}$
2	$\{j_1, j_2, j_3, j_4, j_5\}$
3	$\{p_{11}, p_{13}, p_{15}, p_{22}, p_{24}, p_{31}, p_{33}, p_{35}, t_{11}, t_{12}, t_{13}, t_{21}, t_{22}, t_{23}, t_{31}, t_{32}, t_{33}, r_1, r_2, r_3\}$
4	$\{p_{12}, p_{14}, p_{21}, p_{23}, p_{25}, p_{32}, p_{34}\}$

The solution to class 1 is given in Figure 16 and Table 19.

FIGURE 16

Solution of Example Configuration 3 and Priority Class 1

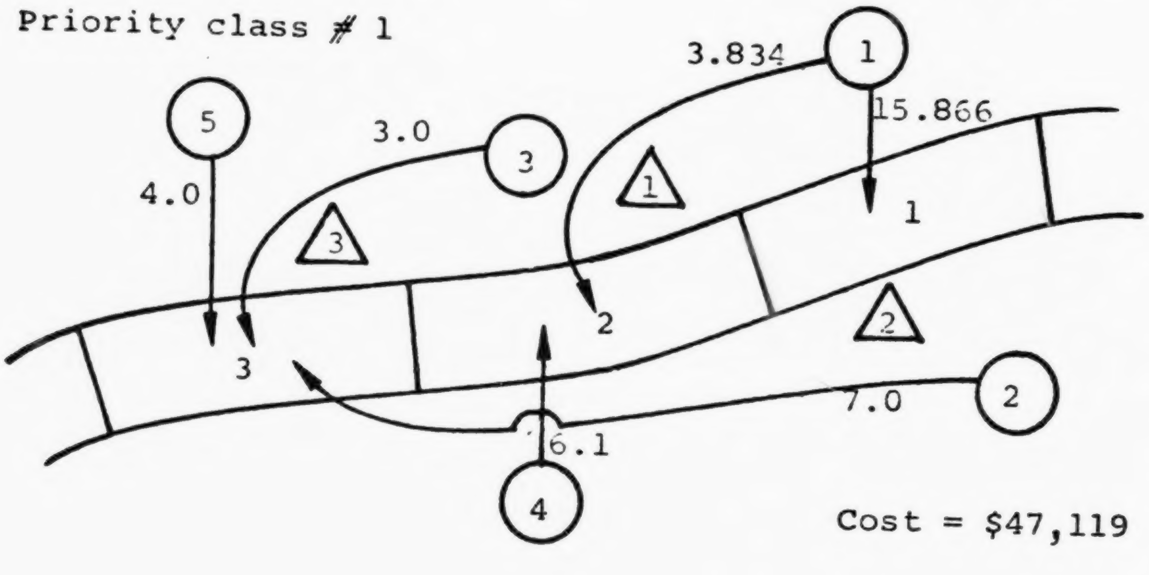


TABLE 19

Solution Data for Example Configuration 3 and Priority Class 1

Polluter Number	Lower Bound lb/MG	Concentration lb/MG	Upper Bound lb/MG	% removal BOD
1	52	155	155	85
2	136	1801	1801	35
3	74	666	666	55
4	93	278	278	85
5	111	557	557	75

This is the pure by-pass piping solution. The annual cost of \$47,119 is considerably better than the annual cost of \$180,843 for pure treatment at the polluter given in Table 16 and the annual cost of \$93,738 for pure treatment at regional plants given in Figure 13 and Table 13. These comparisons however are misleading for general full scale problems. The very short distances obtainable with three sections and the negative goal in section 3 account for the distortion. In general all three treatment modes can be utilized efficiently to contribute to the optimal solution. When seeking the general mixed treatment mode optimum, it is usually best to optimize first in Configuration 1 Priority Class 1. This forces the regional plants to operate at high enough levels to achieve their economies of scale before introducing other cost comparisons.

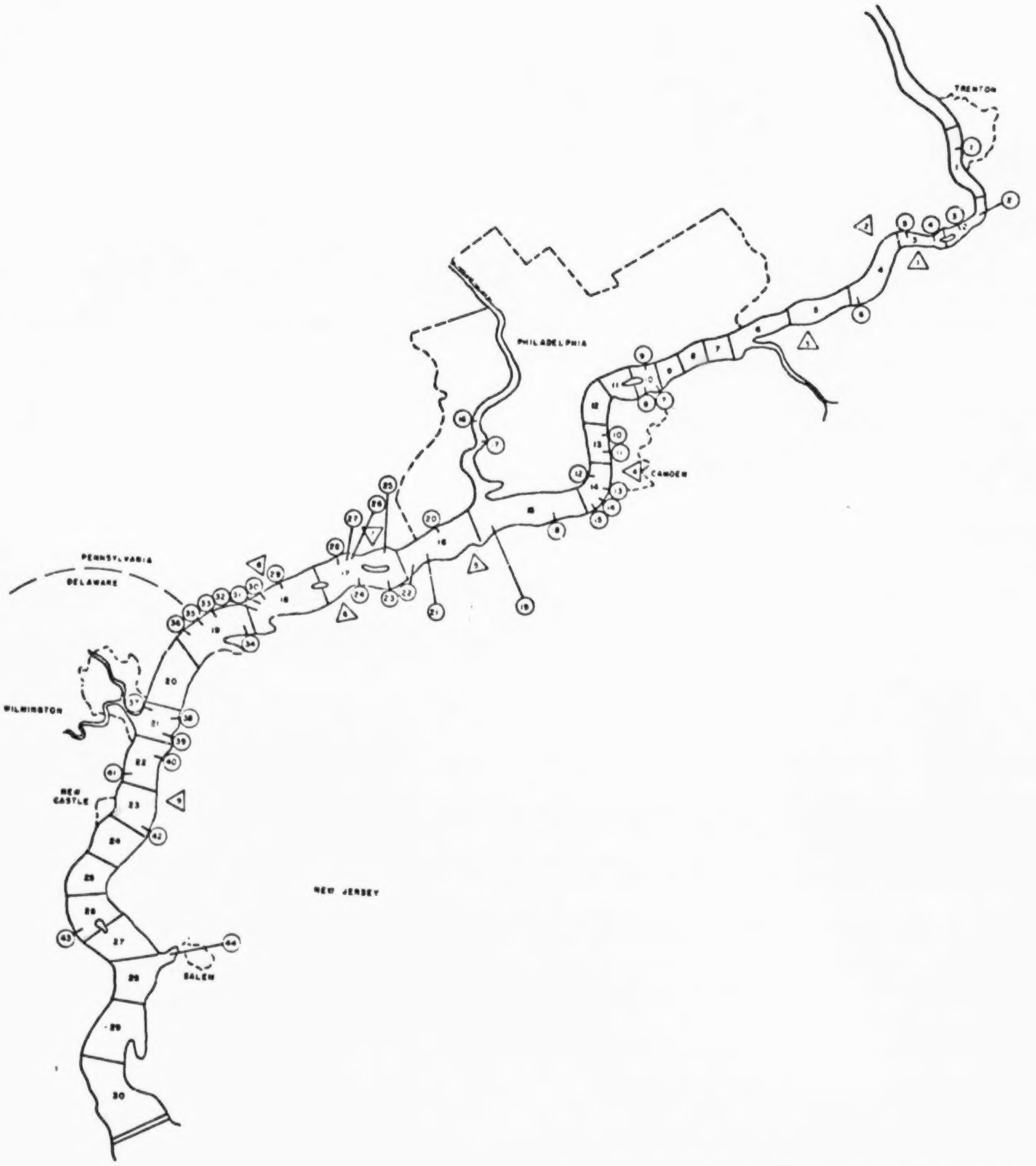
VII. COST ANALYSIS OF WATER POLLUTION CONTROL SCHEMES FOR THE DELAWARE ESTUARY

The Delaware Estuary is bordered by a large metropolitan and industrial complex, including one of the largest oil refining and chemical areas in the United States. Hence a number of substantial and complex wastes enter the estuary along the major part of its 84 mile length. Figure 17 shows the estuary segmented into 30 sections of 10,000 feet or 20,000 feet in length. Also, there are located 44 major waste dischargers and their discharge section along with 9 potential regional treatment plant sites. Note that the influences of the tide extend to Trenton, New Jersey.

A report by Schaumburg [12] gives a detailed comparison of the total costs of different water pollution control schemes for the Delaware Estuary. He took the average dissolved oxygen concentration in the Delaware Estuary in the summer of 1964 as a baseline. Thus given a water quality standard (say 3 mg/1) we calculate a set of D.O. goals by subtraction. A standard of 3 mg/1 means that the D.O. in each of the 30 sections must be at least 3 mg/1. Since he gives extensive results for this standard, we will use it too. The corresponding D.O. goals are given in Table 20.

FIGURE 17

Map of Delaware Estuary


TRENTON
PHILADELPHIA
CAMDEN
PENNSYLVANIA
DELAWARE
WILMINGTON
NEW CASTLE
NEW JERSEY
SALEM

○ = Major Waste Discharger

Δ = Potential Regional Plant Site

TABLE 20

D.O. standard of 3 mg/l

Section	Goal	Section	Goal	Section	Goal
1	-	11	0.7	21	0.1
2	-	12	1.5	22	-
3	-	13	1.8	23	-
4	-	14	1.9	24	-
5	-	15	1.9	25	-
6	-	16	2.0	26	-
7	-	17	2.0	27	-
8	-	18	1.8	28	-
9	-	19	1.6	29	-
10	0.8	20	0.8	30	-

In the next few paragraphs we will give a short description of each of the schemes he considered along with their costs;

1) Required secondary: no specific standard is given and all polluters are required to use a secondary treatment process. Those already using a secondary process do nothing.

2) Uniform treatment: given a set of quality constraints all polluters are required to reduce BOD discharges by the same percentage.

3) Least cost method: identical to treatment at the polluter (see formulation on page 51).

4) 7-zone uniform: the polluters are divided into 7 zones according to their location along the estuary. Within each zone all polluters reduce BOD discharges by the same percentage.

5) 3-zone geographic: the estuary is divided into 3 zones of equal length and within each zone all polluters reduce BOD discharges by the same percentage.

6) 2-zone uniform: zone 1 contains all municipal polluters and zone 2 all industrial polluters. Within each zone all polluters reduce BOD discharges by the same percentage.

7) 3-zone percentage: zone 1 consists of all polluters currently treating over 80%. Zone 2 consists of all polluters treating between 40 and 80% and zone 3 consists of all polluters treating less than 40%. Again within each zone all polluters reduce BOD discharges by the same percentage.

8) Effluent charge: polluters are charged for each pound of BOD discharged into the estuary, and in accordance with marginal cost analysis they reduce discharges until the marginal cost of reducing discharges equals the amount of the effluent charge.

The total annual cost of each of these schemes for a quality standard of 3 mg/l is given in Table 21. Schaumburg gives these costs as a present value over a discounting period of 20 years with a present value factor of 13.

TABLE 21

Cost Comparison of Treatment Schemes

Treatment Scheme	Total Cost ($ x 10^6)
1) required secondary	10.331
2) uniform treatment	8.153
3) least cost	4.0
4) 7-zone uniform	5.138
5) 3-zone geographic	5.715
6) 2-zone uniform	7.084
7) 3-zone percentage	7.292
8) effluent charge	4.692

Uniform treatment and required secondary are favorite schemes of politicians because they are "fair". From Table 21, we see that the cost of these "fair" schemes are "unfair" to whomever pays for them.

Part of the input data for the Delaware Estuary problem is given in Section V (Tables 9 and 10). The transfer matrix is given in Table 22 and the three distance tables are given in Tables 23, 24 and 25. In Table 24, the infinity sign ∞ indicates large distances and effectively rules out piping across the estuary where our piping costs would not be accurate.

We began by solving the linear case of treatment at the polluter. Our solution gave a total annual cost of 4.115 million dollars. We believe the additional $115,000 can be attributed to further refinements made in our transfer matrix. Unfortunately, Schaumburg does not report the transfer matrix, flows, or concentrations which he used. Schaumburg's data as well as ours was supplied by the Delaware Estuary Comprehensive Study Group; and, we feel confident in comparing our results with his results. Also, it appears that he rounded his final total costs. The details of our solution are given in Table 26. In Table 26, the values in column 2 are effluent BOD concentrations at the optimal solution. It is interesting to note that of the 44 polluters on the Delaware Estuary, only 7 have 80% or better BOD removal. From Table 26, we see that 22 (or half) of the polluters have 80% or better BOD removal in the optimal treatment at the polluter solution.

TABLE 22

Transfer Matrix for the Delaware Estuary

	1	2	3	4	5	6	7	8	9	10
1	7.834E-06	9.331E-07	5.767E-08	1.343E-09	3.271E-11	9.865E-13	2.057E-13	8.947E-14	3.855E-14	1.687E-14
2	1.387E-05	1.069E-05	8.356E-07	2.130E-08	5.439E-10	1.685E-11	3.557E-12	1.559E-12	6.758E-13	2.973E-13
3	1.771E-05	1.706E-05	1.240E-05	4.084E-07	1.157E-08	3.778E-10	8.166E-11	3.626E-11	1.589E-11	7.057E-12
4	1.806E-05	1.892E-05	1.770E-05	1.274E-05	4.770E-07	1.738E-08	3.921E-09	1.782E-09	7.953E-10	3.587E-10
5	1.616E-05	1.768E-05	1.834E-05	1.750E-05	1.330E-05	6.325E-07	1.548E-07	7.326E-08	3.369E-08	1.557E-08
6	1.159E-05	1.302E-05	1.428E-05	1.528E-05	1.529E-05	1.227E-05	3.729E-06	1.924E-06	9.372E-07	4.524E-07
7	9.761E-06	1.110E-05	1.248E-05	1.399E-05	1.523E-05	1.519E-05	1.199E-05	7.489E-06	4.044E-06	2.089E-06
8	8.196E-06	9.389E-06	1.070E-05	1.229E-05	1.393E-05	1.510E-05	1.435E-05	1.221E-05	7.409E-06	4.081E-06
9	6.633E-06	7.643E-06	8.808E-06	1.030E-06	1.203E-05	1.378E-06	1.446E-05	1.383E-05	1.177E-05	7.418E-06
10	5.383E-06	6.234E-06	7.251E-06	8.612E-06	1.029E-05	1.227E-05	1.371E-05	1.396E-05	1.356E-05	1.191E-05
11	4.571E-06	5.311E-06	6.217E-06	7.459E-06	9.047E-06	1.105E-05	1.280E-05	1.346E-05	1.386E-05	1.356E-06
12	3.951E-06	4.650E-06	5.469E-06	6.611E-06	8.108E-06	1.008E-06	1.196E-05	1.283E-05	1.369E-06	1.416E-05
13	3.384E-06	3.951E-06	4.664E-06	5.672E-06	7.015E-06	8.833E-06	1.066E-05	1.161E-06	1.267E-05	1.356E-05
14	2.779E-06	3.250E-06	3.848E-06	4.701E-06	5.850E-06	7.437E-06	9.091E-06	9.994E-06	1.108E-05	1.213E-05
15	2.016E-06	2.362E-06	2.806E-06	3.447E-06	4.321E-06	5.555E-06	6.887E-06	7.655E-06	8.630E-06	9.672E-06
16	1.382E-06	1.621E-06	1.931E-06	2.381E-06	3.002E-06	3.891E-06	4.873E-06	5.458E-06	6.224E-06	7.083E-06
17	9.352E-07	1.098E-06	1.311E-06	1.621E-06	2.051E-06	2.672E-06	3.368E-06	3.790E-06	4.353E-06	5.000E-06
18	6.203E-07	7.291E-07	8.710E-07	1.079E-06	1.368E-06	1.789E-06	2.265E-06	2.557E-06	2.950E-06	3.408E-06
19	4.028E-07	4.736E-07	5.663E-07	7.023E-07	8.921E-07	1.169E-06	1.484E-06	1.679E-06	1.943E-06	2.253E-06
20	2.577E-07	3.031E-07	3.625E-07	4.499E-07	5.722E-07	7.509E-07	9.549E-07	1.082E-06	1.254E-06	1.458E-06
21	1.835E-07	2.159E-07	2.583E-07	3.207E-07	4.081E-07	5.359E-07	6.820E-07	7.731E-07	8.970E-07	1.044E-06
22	1.385E-07	1.629E-07	1.950E-07	2.422E-07	3.082E-07	4.049E-07	5.156E-07	5.846E-07	6.788E-07	7.905E-07
23	9.923E-08	1.167E-07	1.397E-07	1.736E-07	2.209E-07	2.904E-07	3.699E-07	4.196E-07	4.873E-07	5.679E-07
24	6.771E-08	7.967E-08	9.537E-08	1.185E-07	1.508E-07	1.983E-07	2.527E-07	2.867E-07	3.332E-07	3.885E-07
25	4.750E-08	5.589E-08	6.691E-08	8.313E-08	1.059E-07	1.392E-07	1.774E-07	2.014E-07	2.341E-07	2.730E-07
26	3.218E-08	3.786E-08	4.533E-08	5.633E-08	7.174E-08	9.436E-08	1.203E-07	1.366E-07	1.588E-07	1.852E-07
27	2.187E-08	2.574E-08	3.081E-08	3.829E-08	4.878E-08	6.417E-08	8.184E-08	9.291E-08	1.080E-07	1.261E-07
28	1.512E-08	1.780E-08	2.131E-08	2.648E-08	3.374E-08	4.439E-08	5.663E-08	6.430E-08	7.479E-08	8.732E-08
29	8.658E-09	1.019E-08	1.220E-08	1.517E-08	1.933E-08	2.544E-08	3.246E-08	3.687E-08	4.290E-08	5.011E-08
30	3.766E-09	4.432E-09	5.308E-09	6.599E-09	8.410E-09	1.107E-08	1.413E-08	1.606E-08	1.869E-08	2.184E-08

Transfer Matrix (Continued)

	11	12	13	14	15	16	17	18	19	20
1	8.916E-15	4.846E-15	2.598E-15	1.423E-15	6.122E-16	2.085E-16	7.202E-17	2.539E-17	9.338E-18	3.625E-18
2	1.578E-13	8.605E-14	4.627E-14	2.540E-14	1.096E-14	3.744E-15	1.295E-15	4.573E-16	1.683E-16	6.537E-17
3	3.771E-12	2.069E-12	1.118E-12	6.161E-13	2.674E-13	9.170E-14	3.182E-14	1.125E-14	4.147E-15	1.612E-15
4	1.938E-10	1.073E-10	5.843E-11	3.241E-11	1.418E-11	4.894E-12	1.705E-12	6.049E-13	2.233E-13	8.693E-14
5	8.555E-09	4.800E-09	2.644E-09	1.480E-09	6.550E-10	2.280E-10	7.992E-11	2.845E-11	1.053E-11	4.107E-12
6	2.557E-07	1.466E-07	8.220E-08	4.666E-08	2.101E-08	7.406E-09	2.617E-09	9.368E-10	3.480E-10	1.360E-10
7	1.229E-06	7.255E-07	4.161E-07	2.403E-07	1.104E-07	3.950E-08	1.409E-08	5.073E-09	1.892E-09	7.413E-10
8	2.486E-06	1.502E-06	8.767E-07	5.128E-07	2.390E-07	8.640E-08	3.101E-08	1.121E-08	4.193E-09	1.646E-09
9	4.810E-06	3.023E-06	1.814E-06	1.082E-06	5.152E-07	1.890E-07	6.842E-08	2.488E-08	9.337E-09	3.673E-09
10	8.634E-06	5.767E-06	3.600E-06	2.205E-06	1.080E-06	4.032E-07	1.476E-07	5.402E-08	2.036E-08	8.031E-09
11	1.213E-06	8.872E-06	5.834E-06	3.690E-06	1.865E-06	7.105E-07	2.630E-07	9.695E-08	3.670E-08	1.452E-08
12	1.382E-05	1.235E-05	8.909E-06	5.930E-06	3.141E-06	1.229E-06	4.620E-07	1.719E-07	6.543E-08	2.597E-08
13	1.389E-05	1.356E-05	1.204E-05	8.782E-06	5.006E-06	2.037E-06	7.815E-07	2.942E-07	1.128E-07	4.500E-08
14	1.278E-05	1.311E-05	1.274E-05	1.130E-05	7.287E-06	3.139E-06	1.238E-06	4.733E-07	1.832E-07	7.349E-08
15	1.049E-05	1.124E-05	1.175E-05	1.180E-05	1.096E-05	5.328E-06	2.214E-06	8.700E-07	3.422E-07	1.386E-07
16	7.821E-06	8.603E-06	9.361E-06	9.965E-06	1.047E-05	9.817E-06	4.854E-06	2.063E-06	8.459E-07	3.508E-07
17	5.580E-06	6.231E-06	6.928E-06	7.593E-06	8.421E-06	9.431E-06	9.108E-06	4.615E-06	2.047E-06	8.839E-07
18	3.829E-06	4.315E-06	4.859E-06	5.415E-06	6.182E-06	7.500E-06	8.643E-06	8.504E-06	4.484E-06	2.088E-06
19	2.542E-06	2.881E-06	3.271E-06	3.681E-06	4.274E-06	5.413E-06	6.747E-06	7.897E-06	7.892E-06	4.340E-06
20	1.649E-06	1.877E-06	2.141E-06	2.424E-06	2.844E-06	3.691E-06	4.794E-06	6.051E-06	7.143E-06	7.221E-06
21	1.182E-06	1.347E-06	1.540E-06	1.749E-06	2.061E-06	2.703E-06	3.568E-06	4.630E-06	5.760E-06	6.569E-06
22	8.960E-07	1.022E-06	1.170E-06	1.331E-06	1.572E-06	2.075E-06	2.766E-06	3.648E-06	4.669E-06	5.640E-06
23	6.440E-07	7.352E-07	8.425E-07	9.597E-07	1.136E-06	1.507E-06	2.025E-06	2.705E-06	3.536E-06	4.446E-06
24	4.408E-07	5.036E-07	5.776E-07	6.588E-07	7.814E-07	1.041E-06	1.408E-06	1.900E-06	2.528E-06	3.278E-06
25	3.099E-07	3.542E-07	4.066E-07	4.641E-07	5.512E-07	7.368E-07	[illegible]	1.361E-06	1.833E-06	2.425E-06
26	2.103E-07	2.406E-07	2.763E-07	3.157E-07	3.754E-07	5.033E-07	6.871E-07	9.407E-07	1.280E-06	1.725E-06
27	1.432E-07	1.639E-07	1.884E-07	2.154E-07	2.565E-07	3.449E-07	4.729E-07	6.517E-07	8.959E-07	1.227E-06
28	9.923E-08	1.136E-07	1.306E-07	1.495E-07	1.782E-07	2.403E-07	3.307E-07	4.584E-07	6.359E-07	8.834E-07
29	5.697E-08	6.526E-08	7.512E-08	8.606E-08	1.028E-07	1.391E-07	1.926E-07	2.693E-07	3.783E-07	5.360E-07
30	2.484E-08	2.847E-08	3.279E-08	3.760E-08	4.497E-08	6.106E-08	8.491E-08	1.195E-07	1.595E-07	2.437E-07

Transfer Matrix (Continued)

	21	22	23	24	25	26	27	28	29	30
1	1.829E-18	1.104E-18	6.589E-19	3.874E-19	2.542E-19	1.644E-19	1.048E-19	6.734E-20	3.316E-20	1.177E-20
2	3.298E-17	1.991E-17	1.189E-17	6.989E-18	4.586E-18	2.966E-18	1.892E-18	1.215E-18	5.983E-19	2.124E-19
3	8.137E-16	4.913E-16	2.934E-16	1.725E-16	1.132E-16	7.323E-17	4.671E-17	3.001E-17	1.478E-17	5.246E-18
4	4.390E-14	2.651E-14	1.584E-14	9.316E-15	6.114E-15	3.955E-15	2.523E-15	1.621E-15	7.983E-16	2.835E-16
5	2.075E-12	1.254E-12	7.495E-13	4.409E-13	2.894E-13	1.873E-13	1.195E-13	7.676E-14	3.781E-14	1.343E-14
6	6.881E-11	4.160E-11	2.487E-11	1.464E-11	9.613E-12	6.220E-12	3.969E-12	2.551E-12	1.257E-12	4.463E-13
7	3.755E-10	2.272E-10	1.359E-10	8.002E-11	5.256E-11	3.402E-11	2.171E-11	1.395E-11	6.875E-12	2.442E-12
8	8.340E-10	5.048E-10	3.021E-10	1.780E-10	1.169E-10	7.567E-11	4.830E-11	3.105E-11	1.530E-11	5.436E-12
9	1.864E-09	1.129E-09	6.759E-10	3.982E-10	2.617E-10	1.694E-10	1.082E-10	6.953E-11	3.427E-11	1.218E-11
10	4.079E-09	2.473E-09	1.481E-09	8.732E-10	5.739E-10	3.717E-10	2.373E-10	1.526E-10	7.523E-11	2.674E-11
11	7.304E-09	4.479E-09	2.685E-09	1.583E-09	1.041E-09	6.743E-10	4.307E-10	2.769E-10	1.336E-10	4.854E-11
12	1.373E-08	8.032E-09	4.818E-09	2.843E-09	1.870E-09	1.212E-09	7.741E-10	4.979E-10	2.456E-10	8.731E-11
13	2.297E-08	1.396E-08	8.382E-09	4.950E-09	3.257E-09	2.111E-09	1.349E-09	8.680E-10	4.283E-10	1.523E-10
14	3.760E-08	2.289E-08	1.376E-08	8.133E-09	5.354E-09	3.472E-09	2.220E-09	1.429E-09	7.052E-10	2.509E-10
15	7.118E-08	4.344E-08	2.616E-08	1.549E-08	1.021E-08	6.623E-09	4.237E-09	2.728E-09	1.348E-09	4.797E-10
16	1.820E-07	1.117E-07	6.758E-08	4.015E-08	2.652E-08	1.724E-08	1.104E-08	7.120E-09	3.523E-09	1.256E-09
17	4.660E-07	2.888E-07	1.759E-07	1.051E-07	6.966E-08	4.541E-08	2.916E-08	1.883E-08	9.342E-09	3.336E-09
18	1.132E-06	7.123E-07	4.390E-07	2.646E-07	1.763E-07	1.154E-07	7.438E-08	4.817E-08	2.399E-08	8.593E-09
19	2.477E-06	1.602E-06	1.007E-06	6.159E-07	4.141E-07	2.730E-07	1.769E-07	1.151E-07	5.767E-08	2.075E-08
20	4.659E-06	3.178E-06	2.074E-06	1.302E-06	8.890E-07	5.931E-07	3.880E-07	2.543E-07	1.287E-07	4.665E-08
21	6.074E-06	4.699E-06	3.290E-06	2.162E-06	1.515E-06	1.029E-06	6.833E-07	4.528E-07	2.326E-07	8.524E-08
22	5.902E-06	5.514E-06	4.204E-06	2.901E-06	2.084E-06	1.442E-06	9.701E-07	6.494E-07	3.379E-07	1.250E-07
23	5.020E-06	5.119E-06	4.747E-06	3.587E-06	2.689E-06	1.913E-06	1.314E-06	8.924E-07	4.731E-07	1.772E-07
24	3.899E-06	4.196E-06	4.283E-06	3.998E-06	3.247E-06	2.425E-06	1.720E-06	1.195E-06	6.513E-07	2.485E-07
25	2.979E-06	3.306E-06	3.543E-06	3.606E-06	3.387E-06	2.722E-06	2.020E-06	1.445E-06	8.143E-07	3.173E-07
26	2.177E-06	2.475E-06	2.750E-06	2.962E-06	3.014E-06	2.866E-06	2.318E-06	1.744E-06	1.036E-06	4.165E-07
27	1.586E-06	1.841E-06	2.105E-06	2.363E-06	2.533E-06	2.636E-06	2.563E-06	2.106E-06	1.356E-06	5.691E-07
28	1.164E-06	1.373E-06	1.605E-06	1.856E-06	2.059E-06	2.259E-06	2.399E-06	2.352E-06	1.720E-06	7.655E-07
29	7.254E-07	8.735E-07	1.048E-06	1.255E-06	1.445E-06	1.672E-06	1.917E-06	2.123E-06	2.247E-06	1.129E-06
30	3.360E-07	4.103E-07	5.009E-07	6.128E-07	7.212E-07	8.587E-07	1.023E-06	1.193E-06	1.416E-06	1.415E-06

TABLE 23

Mileages from Polluters to Sections on the Delaware Estuary

Polluter Number	Estuary Section Number 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	4	8	12	16	20	22	24	26	28	30	32	34	35	38
2	4	0	4	8	12	15	17	18	20	22	23	25	27	29	32
3	4	0	4	8	12	15	17	18	20	22	23	25	27	29	32
4	5	4	0	4	8	11	13	15	17	18	20	22	24	26	29
5	5	4	0	4	8	11	13	15	17	18	20	22	24	26	29
6	8	6	3	0	4	7	9	10	12	14	15	17	19	21	24
7	28	25	21	17	13	9	6	4	2	0	2	4	6	8	11
8	28	25	21	17	13	9	6	4	2	0	2	4	6	8	11
9	28	25	21	17	13	9	6	4	2	0	2	4	6	8	11
10	35	31	27	23	19	15	12	10	8	6	4	2	0	2	5
11	35	31	27	23	19	15	12	10	8	6	4	2	0	2	5
12	37	33	29	25	21	17	14	12	10	8	6	4	2	0	3
13	37	33	29	25	21	17	14	12	10	8	6	4	2	0	3
14	37	33	29	25	21	17	14	12	10	8	6	4	2	0	3
15	37	33	29	25	21	17	14	12	10	8	6	4	2	0	3
16	40	36	32	28	24	20	17	15	13	11	9	7	5	3	0
17	40	36	32	28	24	20	17	15	13	11	9	7	5	3	0
18	40	36	32	28	24	20	17	15	13	11	9	7	5	3	0
19	40	36	32	28	24	20	17	15	13	11	9	7	5	3	0
20	44	40	36	32	28	24	21	19	17	15	13	11	9	7	4
21	44	40	36	32	28	24	21	19	17	15	13	11	9	7	4
22	48	44	40	36	32	28	25	23	21	19	17	15	13	11	8
23	48	44	40	36	32	28	25	23	21	19	17	15	13	11	8
24	48	44	40	36	32	28	25	23	21	19	17	15	13	11	8
25	48	44	40	36	32	28	25	23	21	19	17	15	13	11	8
26	48	44	40	36	32	28	25	23	21	19	17	15	13	11	8
27	48	44	40	36	32	28	25	23	21	19	17	15	13	11	8
28	48	44	40	36	32	28	25	23	21	19	17	15	13	11	8
29	52	48	44	40	36	32	29	27	25	23	21	19	17	15	12
30	52	48	44	40	36	32	29	27	25	23	21	19	17	15	12
31	56	52	48	44	40	36	33	31	29	27	25	23	21	19	16
32	56	52	48	44	40	36	33	31	29	27	25	23	21	19	16
33	56	52	48	44	40	36	33	31	29	27	25	23	21	19	16
34	56	52	48	44	40	36	33	31	29	27	25	23	21	19	16
35	56	52	48	44	40	36	33	31	29	27	25	23	21	19	16
36	56	52	48	44	40	36	33	31	29	27	25	23	21	19	16
37	63	59	55	51	47	43	40	38	36	34	32	30	28	26	23
38	63	59	55	51	47	43	40	38	36	34	32	30	28	26	23
39	63	59	55	51	47	43	40	38	36	34	32	30	28	26	23
40	65	61	57	53	49	45	42	40	38	36	34	32	30	28	25
41	65	61	57	53	49	45	42	40	38	36	34	32	30	28	25
42	67	63	59	55	51	47	44	42	40	38	36	34	32	30	27
43	73	69	65	61	57	53	50	48	46	44	42	40	38	36	33
44	78	74	70	66	62	58	55	53	51	49	47	45	43	41	37

TABLE 23 CON'T.

Mileages from Polluters to Sections on the Delaware Estuary

	Estuary Section Numbers														
Polluter Number	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	42	46	50	54	57	60	62	64	66	68	68	68	69	71	74
2	34	38	42	46	50	52	53	55	57	59	59	59	60	62	65
3	36	40	44	47	50	53	55	57	59	61	62	64	66	69	72
4	33	37	41	45	49	52	53	55	57	59	60	62	64	67	71
5	33	37	41	45	49	52	53	55	57	59	60	62	64	67	71
6	26	32	34	38	42	44	45	47	49	51	51	51	52	54	57
7	15	19	23	27	30	31	33	34	36	38	39	39	39	39	42
8	15	19	23	27	30	31	33	34	36	38	39	39	39	39	42
9	15	19	23	27	31	34	36	38	40	42	44	46	48	51	55
10	9	13	17	21	23	25	27	29	31	33	34	34	35	38	41
11	9	13	17	21	23	25	27	29	31	33	34	34	35	38	41
12	7	11	15	19	23	25	28	30	32	34	36	38	40	43	47
13	7	11	15	19	21	23	25	27	29	31	32	32	33	36	39
14	7	11	15	19	21	23	25	27	29	31	32	32	33	36	39
15	7	11	15	19	21	23	25	27	29	31	32	32	33	36	39
16	4	7	12	16	20	23	25	27	29	31	33	35	37	40	44
17	4	7	12	16	20	23	25	27	29	31	33	35	37	40	44
18	4	8	12	16	18	20	22	24	26	28	29	29	30	33	36
19	4	8	12	16	18	20	22	24	26	28	29	29	30	33	36
20	0	4	8	12	16	18	20	22	24	25	28	30	32	35	39
21	0	4	8	12	14	17	19	21	23	25	25	25	26	29	32
22	0	2	4	8	10	13	15	17	19	21	21	21	22	25	28
23	4	0	4	8	10	13	15	17	19	21	21	21	22	25	28
24	4	0	4	8	10	13	15	17	19	21	21	21	22	25	28
25	4	0	4	8	12	14	16	18	20	22	24	26	28	31	35
26	4	0	4	8	12	14	16	18	20	22	24	26	28	31	35
27	4	0	4	8	12	14	16	18	20	22	24	26	28	31	35
28	4	0	4	8	12	14	16	18	20	22	24	26	28	31	35
29	8	4	0	4	8	10	12	14	16	18	20	22	24	27	31
30	8	4	0	4	8	10	12	14	16	18	20	22	24	27	31
31	12	8	0	2	4	6	8	10	12	14	16	18	20	23	27
32	12	8	0	2	4	6	8	10	12	14	16	18	20	23	27
33	12	8	4	0	4	6	8	10	12	14	16	18	20	23	27
34	12	8	4	0	4	6	8	10	12	14	16	16	18	20	23
35	12	8	4	0	4	6	8	10	12	14	16	18	20	23	27
36	12	8	4	0	4	6	8	10	12	14	16	18	20	23	27
37	19	15	11	7	3	0	2	4	6	8	9	11	13	16	20
38	19	15	11	7	3	0	2	4	6	8	9	10	11	15	18
39	19	15	11	7	3	0	2	4	6	8	9	10	11	15	18
40	21	17	13	9	5	2	0	2	4	6	7	8	9	13	16
41	21	17	13	9	5	2	0	2	4	6	8	10	12	15	19
42	23	19	15	11	7	4	2	0	2	4	6	6	8	10	14
43	29	25	21	17	13	10	8	6	4	2	0	2	4	7	11
44	33	29	25	21	17	14	12	10	8	6	4	2	0	3	7

TABLE 24

Mileages from Polluters to Regional Plants on the Delaware Estuary

Polluter Number	Regional Plant Number 101	102	103	104	105	106	107	108	109
1	9	∞	18	35	42	48	∞	∞	63
2	5	∞	14	31	38	44	∞	∞	59
3	∞	6	∞	∞	∞	∞	38	45	∞
4	∞	4	∞	∞	∞	∞	36	43	∞
5	∞	2	∞	∞	∞	∞	34	41	∞
6	4	∞	4	23	28	34	∞	∞	49
7	21	∞	11	7	15	21	∞	∞	34
8	21	∞	11	7	15	21	∞	∞	34
9	∞	18	∞	∞	∞	∞	18	25	∞
10	27	∞	17	2	9	15	∞	∞	29
11	27	∞	17	2	9	15	∞	∞	29
12	∞	26	∞	∞	∞	∞	10	17	∞
13	29	∞	19	1	7	13	∞	∞	27
14	30	∞	20	2	6	12	∞	∞	26
15	31	∞	21	3	5	11	∞	∞	25
16	∞	33	∞	∞	∞	∞	6	13	∞
17	∞	32	∞	∞	∞	∞	7	14	∞
18	32	∞	22	3	4	10	∞	∞	24
19	32	∞	22	5	3	8	∞	∞	21
20	∞	25	∞	∞	∞	∞	3	10	∞
21	36	∞	26	7	3	4	∞	∞	18
22	38	∞	28	10	3	3	∞	∞	18
23	40	∞	30	11	4	2	∞	∞	17
24	41	∞	31	12	5	1	∞	∞	16
25	∞	33	∞	∞	∞	∞	2	8	∞
26	∞	32	∞	∞	∞	∞	1	7	∞
27	∞	31	∞	∞	∞	∞	1	6	∞
28	∞	30	∞	∞	∞	∞	2	5	∞
29	∞	33	∞	∞	∞	∞	5	2	∞
30	∞	34	∞	∞	∞	∞	6	1	∞
31	∞	35	∞	∞	∞	∞	7	1	∞
32	∞	36	∞	∞	∞	∞	8	1	∞
33	∞	37	∞	∞	∞	∞	9	2	∞
34	48	∞	38	19	10	5	∞	∞	11
35	∞	38	∞	∞	∞	∞	9	3	∞
36	∞	39	∞	∞	∞	∞	10	4	∞
37	∞	43	∞	∞	∞	∞	14	8	∞
38	55	∞	45	26	17	11	∞	∞	4
39	56	∞	46	28	19	13	∞	∞	2
40	58	∞	48	30	21	15	∞	∞	1
41	∞	47	∞	∞	∞	∞	18	12	∞
42	61	∞	51	33	24	19	∞	∞	2
43	∞	55	∞	∞	∞	∞	26	20	∞
44	69	∞	59	24	23	19	∞	∞	7

TABLE 25

Mileages from Regional Plants to Sections on the Delaware Estuary

Section Number	Regional Plant Number								
	101	102	103	104	105	106	107	108	109
1	8	9	12	37	44	48	48	52	67
2	4	8	10	33	40	44	44	48	63
3	0	4	7	29	36	40	40	44	59
4	4	0	4	25	32	36	36	40	55
5	8	4	0	21	28	32	32	36	51
6	12	7	3	17	24	28	28	32	47
7	15	9	5	14	21	25	25	29	44
8	17	11	6	12	19	23	23	27	42
9	18	13	8	10	17	21	21	25	40
10	20	14	10	8	15	19	19	23	38
11	22	16	11	6	13	17	17	21	36
12	23	18	13	4	11	15	15	19	34
13	25	20	15	2	9	13	13	17	32
14	27	22	17	0	7	11	11	15	30
15	29	25	20	3	4	8	8	12	27
16	32	29	22	7	0	4	4	8	23
17	34	33	28	11	4	0	0	4	19
18	38	37	30	15	8	4	4	0	15
19	42	41	34	19	12	8	8	4	11
20	46	44	38	21	14	10	12	8	7
21	50	48	40	23	17	13	14	10	4
22	52	49	41	25	19	15	16	12	2
23	53	51	43	27	21	17	18	14	0
24	55	53	45	29	23	19	20	16	2
25	57	55	47	31	25	21	22	18	4
26	59	56	47	32	25	21	24	20	6
27	59	58	47	32	25	21	26	22	6
28	59	60	48	33	26	22	28	24	8
29	60	63	50	36	29	25	31	27	10
30	62	67	53	39	32	28	35	31	14

TABLE 26

Solution for Treatment at the Polluter for the Delaware Estuary

Polluter Number	BOD mg/l	% removal	Polluter Number	BOD mg/L	% removal
1	18.6	85	*23	648	93
2	33.4	85	*24	10.5	87
3	1.3	85	25	32.7	85
4	0.8	0	26	32.3	85
5	82	85	*27	93	80
6	80	55	*28	43	80
*7	104	85	29	62.7	60
8	67	75	*30	41	80
*9	62	85	*31	2.5	57
*10	45	90	32	70.1	35
*11	50	85	*33	36	85
12	190	35	*34	1233	72
*13	126	85	*35	23	82
14	40.3	0	36	10.4	0
15	43.1	70	37	172	20
16	93.4	65	38	171.5	40
17	30.5	35	39	212	0
18	69.9	35	40	128	35
*19	31	85	41	80.2	80
*20	40	85	42	187.1	40
*21	92	80	43	1.1	35
22	79.9	35	44	159.9	40

* polluters that increased levels of treatment

This solution, obtained originally by Thomann and Marks, was until recently the best known solution. However, as we are about to show, the general mixed case gives further cost reductions which are quite significant. Of the 44 polluters located along the Delaware Estuary, 22 treat domestic wastes and 22 treat industrial wastes (see Table 10). We have restricted ourselves by not allowing any mixing of the domestic and industrial wastes. Thus, all the polluters which pipe to regional plants will be domestic waste polluters. This exclusion of the industrial polluters from the regional plants adversely effects the optimal solution that we can reach. However, at the present time we do not have any cost data for treating joint industrial-municipal wastes. Fortunately, the Delaware River Basin Commission has recently been awarded a Research and Development Grant to construct a pilot plant to investigate a chemical-biological treatment process for joint industrial-municipal wastes, capable of attaining at least 88 to 93 percent removal of major pollutants. Design, operating and cost information is to be obtained for a 80 MGD regional treatment complex. When this cost data becomes available, industrial wastes could also be piped to the regional plants without any complications in our model and would undoubtedly improve the final optimal solution. The total annual cost of the optimal solution (for the 9 fixed regional plant locations) is 2.291 million dollars and is the best known solution at this time. The details of the solutions are given in Figure 18 and Table 27.

In Figure 18, we see that polluters 23 and 34, both industrial waste polluters, mix in a common by-pass pipe. This unexpected introduction of by-pass piping into the optimal solution is easily explained. Both polluters 23 and 34 have low flows with tremendously high BOD concentrations. Since piping costs are based strictly on flow we get a lot for our money by piping a low flow (high amount of pollution) a large distance. Since most of the pipelines proposed in Figure 18 are not in heavily populated areas, we feel that our piping costs are reasonably accurate.

FIGURE 18
Optimal Solution

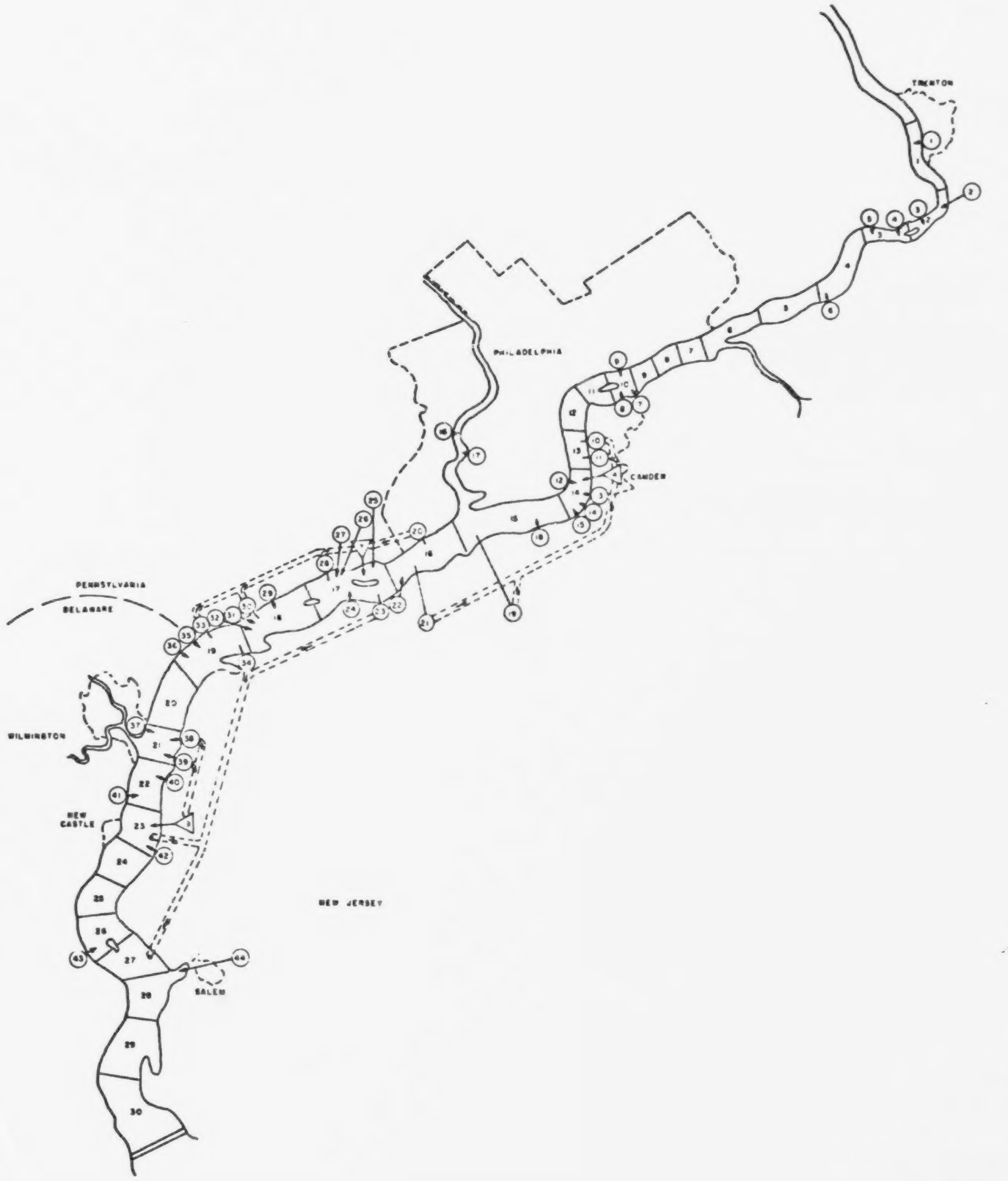

TRENTON
PHILADELPHIA
CAMDEN
PENNSYLVANIA
DELAWARE
WILMINGTON
NEW CASTLE
NEW JERSEY
SALEM

TABLE 27

Data for Optimal Solution

regional plant number	% removal	polluter number	BOD mg/l	% removal
1	-	7	105	85
2	-	9	62	85
3	-	12	155	47
4	70	13	126	85
5	-	24	10.5	87
6	-	27	93	80
7	70	31	2.5	57
8	-	34	1777	60
9	57.9			

Note: The remaining 36 polluters don't increase treatment.

Regional Plant 4:

MGD	
(22.4 + 7.0)	from polluters 10 and 11
(2.0 + 0.7)	from polluters 19 and 21
32.1	discharge to section 14

Regional Plant 7:

MGD	
118.0	from polluter 20
(6.6 + 8.6 + 1.4)	from polluters 28, 30 and 33
134.6	discharge to section 17

Regional Plant 9:

MGD	
(0.902 + 0.467)	from polluters 38 and 39
1.369	discharge to section 23

Note: polluter 38 discharges 0.097 MGD to section 21 and polluter 39 discharges 4.332 MGD to section 21.

By-pass piping:

MGD	
(0.3 + 1.1)	from polluters 23 and 34
0.46514	discharge to section 23
0.93486	discharge to section 27

To complete this section, we intend to make a longhand calculation of the optimal solution to verify that it is correct.

Costs at 8 polluters that treat:

Polluter Number	$ x 10^6	
7	.026359	
9	.127177	
12	.477324	$\frac{199}{13}[170{,}110 - (107.6 \cdot 155 \cdot 8.33)]$
13	.013179	
24	.413668	
27	.013085	
31	.024661	
34	.249282	$\frac{199}{13}[32650 - (1.1 \cdot 1777 \cdot 8.33)]$
	1.344735	

Costs at 3 treatment plants:

Plant 4:

$$r_4 = 0.7$$

$$\bar{v}_4 = 1 - \left[\frac{22.4 \cdot 322 + 7 \cdot 217 + 2 \cdot 124 + 0.7 \cdot 321}{22.4 \cdot 456 + 7 \cdot 333 + 2 \cdot 207 + 0.7 \cdot 458}\right] = .30689$$

$$v_4 = .7 + .30689 - (.7 \times .30689) = .792067$$

$$\text{cost} = .39376\ (32.1)^{0.75}\ (.292067^3 + .19311^3) = .170539$$

Plant 7:

$$r_7 = 0.7$$

$$\bar{v}_7 = 1 - \left[\frac{118 \cdot 159 + 6.6 \cdot 141 + 8.6 \cdot 142 + 1.4 \cdot 156}{118 \cdot 265 + 6.6 \cdot 217 + 8.6 \cdot 203 + 1.4 \cdot 240}\right] = .39247$$

$$v_7 = .7 + .39247 - (.7 \times .39247) = .81774$$

$$\text{cost} = .39376(134.6)^{0.75}\ (.31774^3 + .10753^3) = .518496$$

Plant 9:

$$r_9 = .579$$

$$\bar{v}_9 = 1 - \left[\frac{.902 \cdot 171 + .467 \cdot 212}{.902 \cdot 286 + .467 \cdot 212}\right] = .290579$$

$$v_9 = .579 + .290579 - (.579 \times .290579) = .701333$$

$$\text{cost} = .39376\ (1.369)^{0.75}\ (.201333^3 + .209421^3) = .008644$$

Consolidation of pipes:

Plant 4:
$$\left.\begin{aligned} 1865 \times 2 \times 29.4^{.598} &= .028169 \\ 1865 \times 2 \times 0.7^{.598} &= .003013 \\ 1865 \times 5 \times 2.7^{.598} &= .016889 \end{aligned}\right\}\quad .048071$$

Plant 7:
$$\left.\begin{aligned} 1865 \times 3 \times 118^{.598} &= .097003 \\ 1865 \times 3 \times 1.4^{.598} &= .006842 \\ 1865 \times 4 \times 10^{.598} &= .029562 \\ 1865 \times 2 \times 16.6^{.598} &= .020013 \end{aligned}\right\}\quad .153420$$

Plant 9:
$$\left.\begin{aligned} 1865 \times 2 \times .902^{.598} &= .003506 \\ 1865 \times 2 \times 1.369^{.598} &= .004500 \end{aligned}\right\}\quad .008006$$

By-pass piping:
$$\left.\begin{aligned} 1865 \times 7 \times .3^{.598} &= .006354 \\ 1865 \times 10 \times 1.4^{.598} &= .022806 \\ 1865 \times 6 \times .93486^{.598} &= .010748 \end{aligned}\right\}\quad .039908$$

TOTAL = 2.291819

Referring back to Schaumburg's [12] cost comparisons in Table 21, we see that this total of 2.291819 million annual cost is only about half the previously best known solution of 4 million annual cost for pure treatment at the polluter. It is only about a fourth of the 8.153 million annual cost of "uniform treatment" the currently favored policy. Even allowing for some crudeness in our cost data and transfer coefficients, it would seem evident from this result that Interstate Agencies administering regional treatment complexes could substantially reduce the enormous costs involved in maintaining and improving water quality.

The present mathematical model also provides an excellent planning tool for evaluating the relative costs of alternate water quality goals. It should help establish more realistic goals and development plans. In future work, we shall concentrate on incorporating projected future demand and larger numbers of alternate plant sites with a view to establishing long term time dependent development plans. We shall undoubtedly also have to incorporate into our procedure a periodic updating of the transfer coefficients to account for any substantial alterations in estuary flow caused by the use of by-pass piping or regional treatment plants.

VIII. ECONOMIC CONSIDERATIONS

"Preliminary Ideas"

The economic implications of regional water pollution control systems lead to some very difficult questions. For example, who will pay for the regional treatment plants? Do new polluters wishing to locate along the Delaware have the same right to pollute as polluters already established? Can this right to pollute be taken away or can a polluter be penalized by being made to pay for some part of a regional treatment system while other polluters pay nothing? We do not intend to resolve any of these issues at this time. However, we sketch out here an approach to implementing an efficient regional treatment system to show the relevance of the results of section VII. The problem is viewed here in terms of collective or joint decision making between the public which desires better water quality and expresses this via governmental agencies and the polluters or users of the water resource. The latter may be roughly broken down into industrial and municipal users. The issues involved in determining the "true" goals of various groups competing for the resource are complex since these groups may overlap.

Decision making in a democratic society is often a difficult process. It is well-known that there are problems involved in finding a satisfactory voting mechanism to arrive at agreement and the determination of a concensus is even more complex. Suppose governmental decision makers must decide on a suitable system of treatment and charges based on water quality standards. One possible plan would be to ask each group involved for their proposal on what water quality should be and how to achieve it. Each proposal could then be evaluated to determine its costs (both social and direct). This plan could become a highly expensive exercise and would perhaps not yield satisfactory proposals since groups often have very hazy views on the social implications of their positions. Thus the effective use of sophisticated mathematical techniques and models which are reasonable approximations to reality may be a critical ingredient to efficient decision making in a democratic society. Without this available, groups may be making choices in almost complete ignorance of their implication or the decision makers may be forced into a quasi-dictatorial situation where only one or two alternatives can be examined. By building a model representing the various decision making units, industrial and municipal waste dischargers and using goals specific to various regions of the estuary, we can know the rough implications of any proposal in a micro-sense in contrast to the cost benefit analysis where some mystic aggregated figures are conjured up to count the total costs and total benefits. Even ignoring the somewhat crude measures used in cost benefit studies, the usefulness of an aggregation of that type as a basis for decision making has been highly overrated. If decisions are to be made on the basis of a reconciliation of the positions of various groups, each one must be able to judge the benefits and effects of any proposal on its own situation.

The most popular scheme for the elimination of water pollution is the effluent charge. The argument usually made is that water quality is a good that is being used up by polluters and should be priced as any other good. The price is called an effluent charge. Setting water quality standards in effect fixes the amount of water quality for sale. One proposed way of determining the correct effluent charge is by an iterative process. Some measure of water quality utilization is first defined as a basis for pricing. Prices are set at an arbitrary level for the last section of the waterway. For a given pricing scheme if the amount of water quality used in some section is greater than that available, the price is increased; if more is available than required, the price is decreased. Assuming convergence of this process we end with a situation that not only meets the goals set forth but does so in a manner that minimizes the total waste treatment costs for industry in that region. The approach has a certain appeal for the following reasons.

1. As a starting procedure cost estimates of pollution control can be made to determine the approximate range for correct prices. Furthermore by using mathematical programming, dual prices are obtained which can be used to judge how far from optimal the estimated prices are.
2. The effluent charge system is efficient in the sense that other approaches such as uniform treatment will cause total waste treatment costs to be higher.
3. The prices give correct incentives to the polluters. A polluter may reduce his costs by discovering alternate products or processes.

What are some of the objections to an effluent charge approach?

1. The concept of auctioning off the water quality resource has some defects. In a static model of firm

behavior an equilibrium price is determined and then a supply of the good is produced in response. In reality, behavior is dynamic. There is the possibility of sudden surges or peaks of demand perhaps resulting from the variability of production processes. Here there is no limitation on the amount of the water resource used by a firm; it may be purchased until total exhaustion of the water quality resource is achieved. The problem is we are trying to superimpose a technique that has validity in a static situation on a highly dynamic phenomenon.

2. It may be to the interests of the polluters to mislead the regional authority as to their actual treatment costs. By indicating that treatment costs are very high a low effluent charge would be set. There may be a considerable institutional lag before the charge can be reset to a higher value.
3. The validity of the static pricing allocation model is doubtful. The requirement for convexity is probably not met.
4. There is a substantial equity issue. Assuming a charge scheme were to work correctly, firms might incur substantial cost differences. For this reason political objections to the effluent proposal may be considerable.

With an ordinary good there can be no justification for refusing to pay a price. The case of water pollution is somewhat different since originally water quality was not priced.

The good was available and served as an inducement for industries to locate on the waterway. Implicitly a right was given to use whatever water quality was available. With effluent charges there is no cognizance of these prior rights. A new factory would be subject to the same type of effluent charge as earlier polluters.

As indicated earlier we shall approach the problem of a viable regional treatment system in terms of a collective decision making problem. The solutions to the mathematical model will provide guidelines and implication of various situations. We shall use a combination of the following:

1. Effluent standards for each polluter.
2. Marginal effluent charges (payments) to induce appropriate reductions in discharge.
3. Taxation on polluters based on the benefits derived from the pollution services of the regional authority. In the present study we consider the benefits of regional plants and piping.
4. The use of licenses that are saleable in order to reallocate effluent standards and rights to utilize the regional treatment facilities.

For the proposed implementation of the regional treatment system on the Delaware we indicate how a specific financing scheme could be developed.

(To Be Filled In)

Cost Data For
Financing Regional Treatment System

Polluters	1	2	3
1			
2			
3			
-			
-			

1 Costs to each polluter without regional treatment in order to meet the standard (i.e., costs to polluters using treatment at polluter only obtained from optimization - should total 4 million annual cost)
2 Costs to each polluter with regional treatment
3 Net Savings - difference of 1 and 2

The total of column 3 should be positive reflecting the savings obtainable with regional treatment. Let α be the factor which reduces the total of 3 to the cost of the regional treatment system. This factor is the proportion of net savings that each polluter must contribute to the cost of the regional treatment system. Note that a polluter may contribute to regional treatment even though he does not actually use the regional treatment plant since under the effluent standard he benefits by being able to discharge a greater amount of effluent than without regional treatment.

In return for payment of the benefit tax polluters are allocated allowable amounts of effluent discharge. These amounts are listed below:

(To Be Filled In)

Polluters	Discharge Amounts

Some of the polluters, because of high costs of treatment at their own plant, use the regional plant. The benefit tax covers their use of this facility.

The right to discharge wastes and to use the regional treatment plant will be considered as saleable goods. The regional authority sets a limit purchase price based on the value of the dual variables.

(To Be Filled In)

Polluter	Dual Variable Effluent Discharge	Dual Variable Regional Plant

The granting of transferable property rights may be an efficient method of coordinating different standards and use of the regional treatment system. For a given quality goal the model determines effluent discharge standards. Each of the present polluters may be issued a license that conveys a right to discharge an amount of waste not to exceed the effluent standard. This right may be given for a fixed number of years. However during that time the polluter would have the right to sell off part of all of his rights to pollute. For example, a new plant finding the Delaware an exceptionally desirable location would have the possibility of purchasing pounds of BOD from existing polluters. The possibility of this sale would induce present dischargers to find process changes or treatment economies which would allow for a profitable sale. Over time the use of the stream for waste disposal is allocated to the users deriving the most benefit. A useful byproduct of having a market in waste discharge rights would be the possibility of valuing the stream among different uses. License fees for fishing and other recreational uses may be measured against the total market value of pollution rights and the relative movements of these numbers may at least give some aid in resetting the quality goals.

License fees may also be used to finance regional treatment plants. According to the solution of the model polluters are allocated licenses to use the treatment plant based on the polluter's cost of waste treatment. After the inital allocation has been made, any other plant wishing to make use of the regional treatment facility would have to purchase a license from a current user for the BOD amount and pay for piping. Only in the case where the polluter purchasing the license would have greater savings than a current user would there be a sale. Thus with license fees we obtain from among the groups of potential users the ones with the greatest savings from use of the regional facility. Note that all users may pay for piping connections to the plant.

APPENDIX A

(Nomenclature)

Section II and the notation in that section are independent of the rest of the report. The meaning of symbols used in Section II apply only to that section while the symbols defined below are used throughout the remainder of the report.

a_{ij}	the change in DO in section i of the estuary due to a change in input of BOD in section j of the estuary, mg/l/lb/day where $a_{ij} \in A$
B_i	the maximum amount of BOD that can be removed for a particular s_i at polluter i, lb/day
c_i	the required change in DO in section i, mg/l where $c_i \in C$
d^1_{ij}	the distance from polluter j to the center of section i, miles
d^2_{ij}	the distance from polluter j to regional treatment plant i, miles
d^3_{ij}	the distance from regional treatment plant j to section i, miles
D^k_{ij}	the piping cost coefficient associated with a particular d^k_{ij}, \$/MGD-yr
f_{ij}	the flow from polluter j to section i, MGD where $f_{ij} \in F$
G^1	the set of material balance at a polluter constraints
G^2	the set of material balance at a regional plant constraints
G^3	the set of quality constraints
j_i	the BOD concentration of effluent from polluter i, lb/MG where $j_i \in J$
$\bar{\bar{j}}_i$	the untreated BOD concentration of influent to polluter i, mg/l

J1, J2, J3, and J4	the BOD concentrations used in the piecewise linear cost function, $lb/MG \cdot 10^{-3}$
$K_{m,n}$	a matrix with all elements zero except for a unit element in the mth row and the nth column
K	the piecewise linear cost function, $\$ x 10^6$
K_c	the cost of BOD removal at a waste water treatment plant $\$ x 10^3/MGD$
k1, k2, and k3	the cost coefficients used in the piecewise linear cost function $\$/lb/MG \cdot 10^{-3}$
p_{ij}	the flow from polluter j to treatment plant i, MGD where $p_{ij} \epsilon P$
Q_{ij}	a flow corresponding to a particular f_{ij}, p_{ij}, or t_{ij} (not necessarily equal), MGD
Q	the design capacity of a waste water treatment plant, MGD
q_i	the fractional removal of BOD at polluter i, dimensionless
r_i	the fractional removal of BOD at regional treatment plant i, dimensionless where $r_i \epsilon R$
s_i	the cost of additional removal of BOD at polluter i, \$/lb/day
t_{ij}	the flow from treatment plant j to section i, MGD where $t_{ij} \epsilon T$
U_i	the BOD discharge from polluter i, lb/day
v_i	the translated fractional removal of BOD at regional treatment plant i dimensionless
w	the fractional removal of BOD at a waste water treatment plant, dimensionless
UB_i	upper bound concentration of polluter i
LB_i	lower bound concentration of polluter i

APPENDIX B

(References)

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1 *Accession Number*

2 *Subject Field & Group*
06 A

SELECTED WATER RESOURCES ABSTRACTS
INPUT TRANSACTION FORM

5 *Organization*
Federal Water Quality Administration

6 *Title*
Mathematical Programming for Regional Water Quality Management

10 *Author(s)*
Glenn W. Graves
Andrew B. Whinston
Gordon B. Hatfield

11 *Date*
8/70

12 *Pages*
123

15 *Contract Number*

16 *Project Number*
16110 FPX

21 *Note*

22 *Citation*
Graduate School of Business Administration
University of California, Los Angeles

23 *Descriptors (Starred First)*
*Regional Analysis, *Waste Treatment, *River Basin Development, *Optimum Development Plans, Estuaries, Simulation

25 *Identifiers (Starred First)*
Nonlinear Programming, Cost Minimization

27 *Abstract*

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WR:102 (REV. OCT. 1968)
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WASHINGTON, D.C. 20240

☆ U. S. GOVERNMENT PRINTING OFFICE : 1970 O - 401-599

1 *Accession Number*
W71-01659

2 *Subject Field & Group*
5G,06A,7C,2L

SELECTED WATER RESOURCES ABSTRACTS
INPUT TRANSACTION FORM

5 *Organization*
California Univ, Los Angeles, Graduate School of Business Administration,

6 *Title*
MATHEMATICAL PROGRAMMING FOR REGIONAL WATER QUALITY MANAGEMENT,

10 *Author(s)*
Graves, Glenn W;
Andrew B Whinston;
Gordon B Hatfield

16 *Project Designation*
FWQA Program 16110 FPX

21 *Note*
PC available from SOD
Order as: I67.13/4;
195850 [redacted] 16110FPX08/70
[redacted], $0.95 in microfiche
PC $1.25

22 *Citation*
Water Pollution Control Research Series, 16110 FPX 08/70, August, 1970. 123p, 27 tab, 18 fig, 13 ref, 2 append,

23 *Descriptors (Starred First)*
*Regional analysis, *Waste water treatment, *River basin development, *Optimum development plan, Estuaries, Simulation, Water quality, Mathematical models, Computer programs, Optimization,

25 *Identifiers (Starred First)*
Nonlinear programming, Cost minimization, *Delaware estuary, Algorithans,

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